A fast algorithm for variational calculations of the Lippmann-Schwinger equation

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Abstract. Existing approaches to the solution of the inverse scattering problems in two and three dimensions depends on linearization of the Helmholtz equation which entails computing the perturbation of the far field due to that of the index of refraction. We present an efficient algorithm for this variational calculation in two dimensions. Our method is based on the merging and splitting procedures already used for the solution of the Lippmann-Schwinger equation [1], [2], [3]. For an M-by-M wavelength problem, the algorithm computes the far field perturbations corresponding to M distinct incident waves in $O(M^3)$ flops.

1 Introduction

The merging and splitting formulae are found essential to the rapid solution of the forward scattering problem [1] governed by the Lippmann-Schwinger integral equation

$$\sigma(x) - \alpha(x) \int_D G(x, \xi)\sigma(\xi)d\xi = \alpha(x)\phi \tag{1}$$

where $\alpha(x) = -k^2 q(x)$, k is the wave number, and q is the scatterer which is compactly supported in a domain $D \subset \mathbb{R}^2$; ϕ is the incident wave, which induces a monopole distribution σ in D that generates the scattered wave

$$\psi(x) = \int_D G(x, \xi)\sigma(\xi), \quad x \in \mathbb{R}^2 \tag{2}$$

The linearization for the inverse scattering problem requires the calculation of variations for the Helmholtz equation, or its Lippmann-Schwinger integral equation. The subject of this paper is rapid computation of the variations using the merging and splitting formalism.

More specifically, let $\delta\alpha$ be the perturbation to α when q is perturbed by δq. There are two problems to solve for the variational calculation of the Helmholtz equation (i) Compute the perturbations to σ and consequently to ψ or its far field, for a given incident wave ϕ to D (ii) Compute the perturbation to the scattering matrix S (see (5) for its definition) when all the incident waves are considered. We will present an algorithm that computes these perturbations, to the first order, in $O(M^3)$ flops for an M-by-M wavelength problem.

Evidently, (1) defines a nonlinear relation between σ and α, or between S and α. Our algorithm is therefore to efficiently compute the matrix-vector multiplication: the Fréchet derivative of the nonlinear map applied to the causal perturbation $\delta\alpha$.

The paper is organized as follows. In §2, we describe the calculation of perturbations, define the scattering matrix and its perturbation, and derive formulae for their direct (as opposed to recursive) computations. In §3, we develop a multiple scattering formalism for the perturbations. §4 and 5 establishes the splitting and merging formulae. Finally in §6, we outline the applications of the formulae to the recursive merging and splitting procedures for the rapid computation of the perturbations.

2 Perturbation of the Scattering matrix

Given α, it follows immediately that the perturbations $\delta\alpha$ and $\delta\sigma$ satisfy, to the first order, the equation

$$\delta\sigma(x) - \alpha(x)\int_D G(x,\xi)\delta\sigma(\xi)d\xi = \alpha(x)\delta\phi + \delta\alpha(x)\left(\phi + \int_D G(x,\xi)\sigma(\xi)\right) \quad (3)$$

Given α, σ can be obtained (efficiently, see [1]) via the solution of (1). Therefore, given α and $\delta\alpha$, the solution of (3) determines $\delta\sigma$. To make this computation efficient, we introduce in this section the scattering matrix S and define its perturbation δS due to $\delta\alpha$.

Given a subdomain Ω of D, denote by $\tilde{\phi}$ the total incident wave to Ω due to the original incident wave ϕ and the scattered wave $\psi_{D\backslash\Omega}$ from the remaining part of D. According to Lemma 3.3,

$$\tilde{\phi} = \phi + \psi_{D\backslash\Omega} \quad (4)$$

Thus the total incident wave to D is identical to ϕ, and is not subject to perturbation $\delta\alpha$. For a proper subdomain Ω of D, however, its total incident wave is subject to perturbation $\delta\alpha$: $\delta\alpha$ gives rise to perturbation to the scattered wave, which leads to perturbation to the total incident wave $\tilde{\phi}$ to Ω. We denote this perturbation by $\delta\tilde{\phi}$.

Following [1], we define the scattering matrix S and its perturbation δS for the subdomain Ω via the formulae

$$\left.\left[\psi, \frac{\partial \psi}{\partial n}\right]\right|_{\partial\Omega} = S\left.\left[\tilde{\phi}, \frac{\partial \tilde{\phi}}{\partial n}\right]\right|_{\partial\Omega} \tag{5}$$

$$\left.\left[\delta\psi, \frac{\partial (\delta\psi)}{\partial n}\right]\right|_{\partial\Omega} = S\left.\left[\delta\tilde{\phi}, \frac{\partial (\delta\tilde{\phi})}{\partial n}\right]\right|_{\partial\Omega} + \delta S\left.\left[\tilde{\phi}, \frac{\partial \tilde{\phi}}{\partial n}\right]\right|_{\partial\Omega} \tag{6}$$

where $\delta\psi$ is the perturbation to the scattered wave generated by $\delta\sigma$

$$\delta\psi(x) = \int_D G(x,\xi)\delta\sigma(\xi), \quad x \in \mathbb{R}^2 \tag{7}$$

Note that (6) results from variational calculation of (5). Let $W(\partial D) = L_2(\partial D) \times L_2(\partial D)$. We will require the four linear operators.

(i) Let $G^{(v,b)} : W(\partial D) \mapsto C^\infty(D)$ be defined by

$$\phi(x) = G^{(v,b)}(u,v) =: \int_{\partial D} \left(v(\xi)G(x,\xi) - u(\xi)\frac{\partial G(x,\xi)}{\partial n(\xi)} \right) ds(\xi), \tag{8}$$

(ii) Let $G^{(v,v)} : L_2(D) \mapsto L_2(D)$ be defined by

$$G^{(v,v)}(\sigma)(x) = \int_D G(x,\xi)\sigma(\xi)d\xi. \tag{9}$$

(iii) Let $P : L_2(D) \mapsto L_2(D)$ be defined by

$$P(\sigma)(x) = \sigma(x) - \alpha(x)\int_D G(x,\xi)\sigma(\xi)d\xi \tag{10}$$

so that (3) can be rewritten

$$P(\sigma) = \alpha\phi, \quad P(\delta\sigma) = \alpha\delta\phi + \delta\alpha\left(1 + G^{(v,v)}P^{-1}\alpha\right)\phi. \tag{11}$$

(iv) $G^{(b,v)} : L_2(D) \mapsto W(\partial D)$ is defined by

$$\left(\psi(x), \frac{\partial\psi(x)}{\partial n}\right) = G^{(b,v)}(\sigma)(x) =: \int_D \left(G(x,\xi), \frac{\partial G(x,\xi)}{\partial n(x)}\right)\sigma(\xi)d\xi, \tag{12}$$

mapping the charge density σ in D to the boundary data of the scattered wave $\psi(x)$ generated by the charges.

It follows immediately from the preceding definition that the scattering matrix S and its perturbation δS can be calculated via the solution of (1) and (3).

Lemma 2.1 *The scattering matrices S, δS for the domain D can be obtained by the solution of (11)*

$$
\begin{aligned}
S &= G^{(b,v)} P^{-1} \alpha G^{(v,b)}, && (13)\\
\delta S &= G^{(b,v)} P^{-1} \delta\alpha (1 + G^{(v,v)} P^{-1} \alpha) G^{(v,b)} && (14)
\end{aligned}
$$

Remark 2.2 *Let $S = S(\alpha)$ be the dependence of S on α. It follows immediately from (13) and (14) that*

$$
S(\alpha + \delta\alpha) - S(\alpha) = \delta S + O(\delta\alpha)^2. \qquad (15)
$$

In other words, δS is the first order term of the perturbation to the scattering matrix which dependes linearly on $\delta\alpha$.

3 Multiple scattering

We derive a multiple scattering formalism for (11) useful for establishing the merging and splitting procedures. The development in this section is analogous to that given in [1], with a notable difference: In [1] we consider multiple scattering among subdomains of D whereas here we consider multiple scattering among subdomains of a subdomain Ω of D, so that it is apt to analyze the perturbation of the total incident wave on Ω which, unlike the total incident wave to D, does not in general vanish.

3.1 Analytical machinery

Let Ω be a subdomain of D, and let there be $m > 1$ disjoint scatterers Ω_i in Ω so that $\Omega = \cup_i \Omega_i$. Denote by S, $\delta S : W(\partial\Omega) \mapsto W(\partial\Omega)$ the scattering matrices and its perturbation of Ω, and by S_i, $\delta S_i : W(\partial\Omega_i) \mapsto W(\partial\Omega_i)$ the scattering matrices and their perturbations of Ω_i. According to Lemma 2.1

$$
\begin{aligned}
S_i &= G_i^{(b,v)} \cdot P_i^{-1} \cdot \alpha_i \cdot G_i^{(v,b)}\\
\delta S_i &= G_i^{(b,v)} \cdot P_i^{-1} \cdot \delta\alpha_i \cdot (1 + G_i^{(v,v)} P_i^{-1} \alpha_i) \cdot G_i^{(v,b)} && (16)
\end{aligned}
$$

We require the three operators to develop the multiple scattering formalism

(1) Restriction $R_i : W(\partial\Omega) \mapsto W(\partial\Omega_i)$, to map the Dirichlet-Neumann data of an incident wave ϕ on $\partial\Omega$ to that on $\partial\Omega_i$

(2) Extension $E_i : W(\partial\Omega_i) \mapsto W(\partial\Omega)$, to map the Dirichlet-Neumann data on $\partial\Omega_i$ of a scattered wave ψ from Ω_i to that on $\partial\Omega$

(3) Translation $T_{ji} : W(\partial\Omega_i) \mapsto W(\partial\Omega_j)$, $i \neq j$, same as E_i except that it maps to the boundary of Ω_j

defined by the formulae

$$
\begin{aligned}
R_i(u,v)(x) &= \int_{\partial\Omega}\left[v(\xi)\left(G,\frac{\partial G}{\partial n(x)}\right)^T - u(\xi)\left(\frac{\partial G(x,\xi)}{\partial n(\xi)},\frac{\partial^2 G(x,\xi)}{\partial n(x)\partial n(\xi)}\right)^T\right]ds(\xi)\\
E_i(u,v)(x) &= -\int_{\partial\Omega_i}\left[v(\xi)\left(G,\frac{\partial G}{\partial n(x)}\right)^T - u(\xi)\left(\frac{\partial G(x,\xi)}{\partial n(\xi)},\frac{\partial^2 G(x,\xi)}{\partial n(x)\partial n(\xi)}\right)^T\right]ds(\xi)\\
T_{ji}(u,v)(x) &= -\int_{\partial\Omega_i}\left[v(\xi)\left(G,\frac{\partial G}{\partial n(x)}\right)^T - u(\xi)\left(\frac{\partial G(x,\xi)}{\partial n(\xi)},\frac{\partial^2 G(x,\xi)}{\partial n(x)\partial n(\xi)}\right)^T\right]ds(\xi)
\end{aligned}
$$

We introduce the variables σ_i, $\delta\sigma_i$, α_i, $\delta\alpha_i$, ϕ_i via the formulae $\sigma_i = \sigma|_{\Omega_i}$, $\delta\sigma_i = \delta\sigma|_{\Omega_i}$, $\alpha_i = \alpha|_{\Omega_i}$, $\delta\alpha_i = \delta\alpha|_{\Omega_i}$, $\phi_i = \phi|_{\Omega_i}$. In the case of two disjoint scatterers, (11) can be reformulated as

$$
\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\phi_1 \\ \alpha_2\phi_2 \end{bmatrix} \tag{17}
$$

and

$$
\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} \delta\sigma_1 \\ \delta\sigma_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\delta\tilde{\phi}_1 + \delta\alpha_1\phi_1 + \delta\alpha_1 G_{11}^{(v,v)}\sigma_1 + \delta\alpha_1 G_{12}^{(v,v)}\sigma_2 \\ \alpha_2\delta\tilde{\phi}_2 + \delta\alpha_2\phi_2 + \delta\alpha_2 G_{22}^{(v,v)}\sigma_2 + \delta\alpha_2 G_{21}^{(v,v)}\sigma_1 \end{bmatrix} \tag{18}
$$

Here, $P_{ji} : L_2(\Omega_i) \mapsto L_2(\Omega_j)$ is defined via

$$
P_{ji}(\sigma)(x) = -\alpha_j(x)\int_{\Omega_i} G(x,\xi)\sigma(\xi)d\xi,\ x \in \Omega_j \tag{19}
$$

and $G_{ji}^{(v,v)} : L_2(\Omega_i) \mapsto L_2(\Omega_j)$ is defined via

$$
G_{ji}^{(v,v)}(\sigma)(x) = \int_{\Omega_i} G(x,\xi)\sigma(\xi)d\xi,\ x \in \Omega_j. \tag{20}
$$

Let ψ_{ji} be the scattered wave from Ω_i which becomes an incident wave in Ω_j and let $\delta\psi_{ji}$ be the scattered wave perturbation from Ω_i which becomes an incident wave perturbation in Ω_j. It follows from (2) that

$$
\psi_{ji}(x) = \int_{\Omega_i} G(x,\xi)\sigma(\xi)d\xi, \quad \delta\psi_{ji}(x) = \int_{\Omega_i} G(x,\xi)\delta\sigma(\xi)d\xi, \quad x \in \Omega_j \tag{21}
$$

Note that (19)–(21) are also valid for $m > 2$ case.

3.2 Multiple scattering between disjoint scatterers

For convenience, we offer an alternative definition of the total incident wave to a subdomain, and verify that it is indeed the one defined by (4).

Definition 3.1 *Let $[\sigma, \delta\sigma] \in L_2(D)$ be the solution of (11). Suppose that there exist $\tilde{\phi}_i \in L_2(\Omega_i)$ and $\delta\tilde{\phi}_i \in L_2(\Omega_i)$ to satisfy*

$$
\begin{aligned}
P_i(\sigma|_{\Omega_i}) &= \alpha_i \tilde{\phi}_i & (22)\\
P_i(\delta\sigma|_{\Omega_i}) &= \alpha_i \delta\tilde{\phi}_i + \delta\alpha_i (1 + G_i^{(v,v)} P_i^{-1} \alpha_i) \tilde{\phi}_i & (23)
\end{aligned}
$$

Then, $\tilde{\phi}_i$ will be referred as to the total incident wave on the scatterer Ω_i, and $\delta\tilde{\phi}_i$ will be referred as its perturbation.

Corollary 3.2 *The total scattered wave ψ induced by ϕ incident on Ω is the superposition of all the scattered waves ψ_i induced by $\tilde{\phi}_i$ incident on Ω_i. The perturbation $\delta\psi$ is the superposition of all the perturbations $\delta\psi_i$*

$$
\psi(x) = \sum_{i=1}^{m} \psi_i(x), \quad \delta\psi(x) = \sum_{i=1}^{m} \delta\psi_i(x), \; x \in \mathbb{R}^2. \tag{24}
$$

Lemma 3.3 *Let ϕ, $\delta\phi$ be the total incident wave and its perturbation on Ω. Then the total incident wave $\tilde{\phi}_i$ on Ω_i is the superposition of ϕ and the scattered wave ψ_{ij} from the other scatterer Ω_i. Likewise, the perturbation $\delta\tilde{\phi}_i$ is the superposition of $\delta\phi$ and $\delta\psi_{ij}$. More precisely,*

$$
\begin{aligned}
\tilde{\phi}_i(x) &= \phi_i(x) + \sum_{j \neq i} \psi_{ij}(x) & (25)\\
\delta\tilde{\phi}_i(x) &= \delta\phi_i(x) + \sum_{j \neq i} \delta\psi_{ij}(x) & (26)
\end{aligned}
$$

for $x \in \Omega_i$.

Proof. We first prove (25) just for $m = 2$ case. Combining (17) and (21) we obtain

$$
\begin{aligned}
P_{11}\sigma_1 &= \alpha_1(x)\phi_1(x) - P_{12}\sigma_2 \\
&= \alpha_1(x) \left[\phi_1(x) + \psi_{12}\right]. & (27)
\end{aligned}
$$

For $j = 1$, (25) follows immediately from (22). Now we prove (26) just for $m = 2$ case. Combining (17), (21), (27) we have

$$
\begin{aligned}
& \delta\alpha_1 \phi_1 + \delta\alpha_1 G_{11}^{(v,v)} \sigma_1 + \delta\alpha_1 G_{12}^{(v,v)} \sigma_2 \\
= \;& \delta\alpha_1 \phi_1 + \delta\alpha_1 \psi_{12} + \delta\alpha_1 G_{11}^{(v,v)} P_{11}^{-1} \alpha_1(x) \left[\phi_1(x) + \psi_{12}\right] \\
= \;& \delta\alpha_1 (1 + G_{11}^{(v,v)} P_{11}^{-1} \alpha_1(x)) \left[\phi_1(x) + \psi_{12}\right]. & (28)
\end{aligned}
$$

Therefore,

$$\begin{aligned} P_{11}\delta\sigma_1 &= -P_{12}\delta\sigma_2 + \alpha_1\delta\tilde{\phi}_1 + \delta\alpha_1\phi_1 + \delta\alpha_1 G_{11}^{(v,v)}\sigma_1 + \delta\alpha_1 G_{12}^{(v,v)}\sigma_2 \\ &= \alpha_1\left[\delta\tilde{\phi}_1 + \delta\psi_{12}\right] + \delta\alpha_1(1 + G_{11}^{(v,v)}P_{11}^{-1}\alpha_1(x))\left[\phi_1(x) + \psi_{12}\right]. \end{aligned} \tag{29}$$

For $j = 1$, (26) follows immediately from (23). □

4 Splitting the incident waves

Splitting is to determine the total incident wave $[\tilde{\phi}_i$ and its perturbation $\delta\tilde{\phi}_i]$ to Ω_i, $i = 1, 2, \ldots, m$, from the total incident wave $[\phi, \delta\phi]$. Given the total incident wave $[\phi, \delta\tilde{\phi}]$ on Ω and the resulting scattered wave $[\psi, \delta\psi]$ from Ω, and given the total incident wave $[\tilde{\phi}_i, \delta\tilde{\phi}_i]$ on Ω_i and the resulting scattered wave $[\psi_i, \delta\psi_i]$ from Ω_i, we denote their Dirichlet and Neumann data pairs on the boundaries $\partial\Omega$ and $\partial\Omega_i$ by

$$\Phi = \left[\phi, \frac{\partial\phi}{\partial n}\right]_{\partial D}, \quad \delta\Phi = \left[\delta\phi, \frac{\partial(\delta\phi)}{\partial n}\right]_{\partial D}, \tag{30}$$

$$\tilde{\Phi}_i = \left[\tilde{\phi}_i, \frac{\partial\tilde{\phi}_i}{\partial n}\right]_{\partial D_i}, \quad \delta\tilde{\mathbf{\Phi}}_i = \left[\delta\tilde{\phi}_i, \frac{\partial(\delta\tilde{\phi}_i)}{\partial n}\right]_{\partial D_i}, \tag{31}$$

$$\Psi = \left[\psi, \frac{\partial\psi}{\partial n}\right]_{\partial D}, \quad \delta\Psi = \left[\delta\psi, \frac{\partial(\delta\psi)}{\partial n}\right]_{\partial D}, \tag{32}$$

$$\Psi_i = \left[\psi_i, \frac{\partial\psi_i}{\partial n}\right]_{\partial D_i}, \quad \delta\Psi_i = \left[\delta\psi_i, \frac{\partial(\delta\psi_i)}{\partial n}\right]_{\partial D_i}. \tag{33}$$

Using the restriction and translation operators, we reformulate (25), (26)

$$\begin{bmatrix} \delta\tilde{\Phi}_i \\ \tilde{\Phi}_i \end{bmatrix} = \begin{bmatrix} R_i\,\delta\tilde{\Phi} \\ R_i\Phi \end{bmatrix} + \begin{bmatrix} \sum_{j\neq i} T_{ij}\,\delta\Psi_j \\ \sum_{j\neq i} T_{ij}\Psi_j \end{bmatrix}. \tag{34}$$

By the definition of the scattering matrix,

$$\Psi_i = S_i\tilde{\Phi}_i, \quad \delta\Psi_i = S_i\,\delta\tilde{\Phi}_i + \delta S_i\Phi_i; \tag{35}$$

therefore,

$$\begin{bmatrix} \delta\tilde{\Phi}_i \\ \Phi_i \end{bmatrix} = \begin{bmatrix} R_i\,\delta\tilde{\Phi} \\ R_i\Phi \end{bmatrix} + \begin{bmatrix} \sum_{j\neq i} T_{ij}(S_j\,\delta\tilde{\Phi}_j + \delta S_j\Phi_j) \\ \sum_{j\neq i} T_{ij}S_j\Phi_j \end{bmatrix}. \tag{36}$$

Definition 4.1 *Let* $W_m = [W(\partial D_1), W(\partial D_2), \ \ldots, \ W(\partial D_m]^T$*, and let the operators*

$$T,\ T_s,\ \delta T_s : W_m \mapsto W_m, \quad R,\ S_p,\ \delta S_p : W(\partial D) \mapsto W_m \tag{37}$$

be defined by the formulae

$$T = \begin{bmatrix} 0 & -T_{12} & \cdots & -T_{1m} \\ -T_{21} & 0 & \cdots & -T_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ -T_{m1} & -T_{m2} & \cdots & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \tag{38}$$

$$T_s = \begin{bmatrix} 0 & -T_{12}S_2 & \cdots & -T_{1m}S_m \\ -T_{21}S_1 & 0 & \cdots & -T_{2m}S_m \\ \vdots & \vdots & \cdots & \vdots \\ -T_{m1}S_1 & -T_{m2}S_2 & \cdots & 0 \end{bmatrix} = T\ diag\{S_i\}, \tag{39}$$

$$\delta T_s = \begin{bmatrix} 0 & -T_{12}\,\delta S_2 & \cdots & -T_{1m}\,\delta S_m \\ -T_{21}\,\delta S_1 & 0 & \cdots & -T_{2m}\,\delta S_m \\ \vdots & \vdots & \cdots & \vdots \\ -T_{m1}\,\delta S_1 & -T_{m2}\,\delta S_2 & \cdots & 0 \end{bmatrix} = T\ diag\{\delta S_i\} \tag{40}$$

$$S_p = (I - T_s)^{-1} R, \qquad \delta S_p = (I - T_s)^{-1}\,\delta T_s\,S_p \tag{41}$$

where $S_p,\ \delta S_p$ *will be referred to as the splitting operators.*

Remark 4.2 *It is easy to verify that* δT_s *and* δS_p *satisfy, in the standard perturbational format, the equations*

$$T_s(\alpha+\delta\alpha) - T_s(\alpha) = \delta T_s + O((\delta\alpha)^2), \quad S_p(\alpha+\delta\alpha) - S_p(\alpha) = \delta S_p + O((\delta\alpha)^2) \tag{42}$$

The preceding definition is motivated by the next theorem which follows immediately from (36).

Theorem 4.3 *(***Splitting the incident wave***) Let* Φ *be the pair of Dirichlet and Neumann data of the total incident wave* ϕ *on* Ω*, and* $\tilde{\Phi}_i$ *be the Dirichlet and Neumann data of the total incident wave* $\tilde{\phi}_i$ *on* Ω_i*. Then*

$$\begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \vdots \\ \tilde{\Phi}_m \end{bmatrix} = S_p \cdot \Phi, \qquad \begin{bmatrix} \delta\tilde{\Phi}_1 \\ \delta\tilde{\Phi}_2 \\ \vdots \\ \delta\tilde{\Phi}_m \end{bmatrix} = [\ S_p,\ \delta S_p\ ] \begin{bmatrix} \delta\tilde{\Phi} \\ \Phi \end{bmatrix}, \tag{43}$$

5 Merging the scattering matrices

In this section, we present formulae for merging disjoint scatterers Ω_i; namely, to calculate the scattering matrices $S,\ \delta S$ of Ω from the scattering matrices $S_i,\ \delta S_i$ of Ω_i.

Theorem 5.1 (**Merge scattering matrices**) *Given the scattering matrices S_i, δS_i, $i = 1, 2, \ldots, m$ of the disjoint scatterers $\{\Omega_i\}$, the scattering matrices S, δS of $\Omega = \cup_i \Omega_i$ can be calculated via the formulae*

$$S = [E_1,\ E_2,\ \cdots,\ E_m]\, diag\{S_i\}\, S_p \tag{44}$$

$$\delta S = [E_1,\ E_2,\ \cdots,\ E_m] \Big[I - diag\{S_i\}\, T \Big]^{-1} diag\{\delta S_i\}\, S_p \tag{45}$$

Proof. We only give proof to (45); the proof of (44) is analogous and can be done independently of (45). It follows from (35) and Theorem 4.3 that

$$\begin{aligned}
\delta\Psi &= \sum_{i=1}^{m} E_i \left(S_i\, \delta\tilde{\Phi}_i + \delta S_i\, \Phi_i \right) \\
&= [E_1,\ E_2,\ \cdots,\ E_m]\, [\mathrm{diag}\{S_i\}\,,\ \mathrm{diag}\{\delta S_i\}] \begin{bmatrix} S_p & \delta S_p \\ 0 & S_p \end{bmatrix} \begin{bmatrix} \delta\tilde{\Phi} \\ \Phi \end{bmatrix} \\
&= [E_1,\ E_2,\ \cdots,\ E_m]\, \mathrm{diag}\{S_i\}\ S_p\, \delta\tilde{\Phi} \\
&\quad + [E_1,\ E_2,\ \cdots,\ E_m] \Big[\mathrm{diag}\{S_i\}\ \delta S_p + \mathrm{diag}\{\delta S_i\}\ S_p \Big] \Phi
\end{aligned} \tag{46}$$

for arbitrary $\Phi, \delta\tilde{\Phi}$. The two terms in the last square bracket can be further simplified using (41)–(39) and the Sherman-Morrison formula

$$\begin{aligned}
&\mathrm{diag}\{\delta S_i\}\, S_p + \mathrm{diag}\{S_i\}\, \delta S_p \\
&= \Big[I + \mathrm{diag}\{S_i\} \left(I - T\, \mathrm{diag}\{S_i\} \right)^{-1} T \Big]\, \mathrm{diag}\{\delta S_i\}\, S_p \\
&= \Big[I - \mathrm{diag}\{S_i\}\, T \Big]^{-1} \mathrm{diag}\{\delta S_i\}\, S_p
\end{aligned}$$

Now (45) follows immediately from (46), (5) and (6). $\square$

6 Conclusions

We have established the merging and splitting formulae for the rapid evaluation of perturbations for the Helmholtz equation. More precisely the merging formula

(44) enables us to recursively compute, in $O(M^3)$ flops, the scattering matrices S for D of M-by-M wavelengths. This process is the main subject of [1]; see it for details of the recursion and its sustaining hierarchy of subdomains.

The merging formula (45) then can be used in identical, recursive manner to compute the perturbation δS for D. In fact, we obtain S and δS for all the subdomains of the hierarchy as we recursively merge. Thus we have accomplished one of the two tasks in the perturbational calculation: The calculation of δS.

The other task, the perturbations to σ and consequently to ψ or its far field, can be completed via the splitting formulae. More specifically, the splitting procedure [1] based on the first formula of (43) reduce the computation of σ in D recursively to that in smaller subdomains of D in $O(M^2 \log(M))$ flops for a given incident wave. A splitting procedure based on the second formula of (43) will recursively compute $\delta\sigma$ in D, as the solution to (3), in $O(M^2 \log(M))$ flops for the same incident wave.

The two perturbational calculations for δS and $\delta\sigma$ were implemented in Fortran. We will report the numerical treatment, mainly the underlying quadrature issues for singular functions, and the performance in a separate paper.

References

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