Local solvability of nonlinear partial differential equations of real principal type

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Introduction

Many problems in differential geometry and in mathematical physics reduce to solving nonlinear (systems) of partial differential equations. If a nonlinear PDE is elliptic, hyperbolic, or parabolic, its solvability in the small follows in a (by now) straightforward way from estimates on the solutions of the linearized PDE. However, a growing number of applications do not fall in one of these classes, so a more general local solvability theorem for nonlinear PDE's is needed. Local solvability of a linear PDE of real principal type was first proved by L. Hörmander. We shall prove local solvability of nonlinear PDE's of real principal type (see §1 for the definition).

Using the Nash-Moser implicit function theorem, the proof reduces to proving so-called Moser-type estimates for solutions to linear PDE's of real principal type. One extremely tedious approach would be a detailed examination of Duistermaat-Hörmander's microlocal Fourier integral operator construction of a parametrix to a linear differential operator. Much of this can be avoided by using two devices. The first involves the choice of a "canonical problem". Duistermaat and Hörmander show that any linear differential operator of real principal type is microlocally either elliptic or conjugate, via Fourier integral operators, to the canonical operator $\partial/\partial x^n$. Since both elliptic operators and $\partial/\partial x^n$ are invertible, this shows that the differential operator is microlocally invertible. We show, instead, that given any differential operator P_0 of strong real principal type (see §1 for the definition), any differential operator P obtained from P_0 by a sufficiently small perturbation is conjugate, via global Fourier integral operators, to P_0. This avoids microlocalization and allows us to use a simpler and more explicit form of Fourier integral operators which we call "classical" (see §B.3). The second device is a painless induction proof of Moser-type estimates (see §A.5 and §B.4)

We will also describe here the following applications of the result:

(1) It is well-known that the study of transonic flow involves the study of a nonlinear Tricomi equation which is elliptic in the subsonic region and hyperbolic in the supersonic region; see [CF] for a discussion.

(2) Recent work of Bryant-Griffiths-Yang shows that the problem of isometrically embedding a Riemannian manifold in Euclidean space often reduces to a system of PDE's of real principal type.

(3) DeTurck-Yang have shown that solving for a Riemannian metric on a 3-manifold with prescribed Riemann curvature tensor (all indices down) also requires solving a system of real principal type.

(4) The Moser-type estimates proved here can be used to generalize a local solvability theorem proved in [Y1] for a certain class of linear overdetermined systems of PDE's to the corresponding class of nonlinear overdetermined systems. This represents a tiny bit of progress towards proving a "C^∞ Cartan-Kähler theorem".

Solving a nonlinear PDE locally can be done in two steps. First, an approximate solution is constructed in a neighborhood of a given point using a formal power series. The formal power series is obtained by applying either the Cauchy-Kovalevskaia or Cartan-Kähler theorems. The second and more difficult step is to use an implicit function theorem to perturb the approximate solution into a true solution. For the standard types of PDE's: elliptic, hyperbolic, and parabolic, the standard implicit function theorem for smooth maps between Banach spaces and a judicious choice of norms yield the desired result. The key step lies in defining an inverse to the linearized operator and proving that it is a bounded operator.

For a PDE of real principal type, an inverse can be obtained from the parametrix constructed by Duistermaat-Hörmander. The problem lies in the fact that the inverse does not "regain" all the derivatives "lost" by the differential operator, so the standard implicit function theorem cannot be used. Although hyperbolic equations suffer the same problem when Sobolev norms are used, the existence of a special "time" co-ordinate allows the use of special norms in which there is no loss of regularity. The Nash-Moser implicit function theorem was devised by Nash to cope exactly with the loss of regularity by the inverse to a differential operator. The price one pays, however, is that the inverse must satisfy so-called Moser-type estimates.

In the appendices there is a short exposition on the Nash-Moser implicit function theorem and one on the estimates and the symbol calculus for pseudodifferential and Fourier integral operators. The key to using the Nash-Moser theorem lies in proving Moser-type estimates, also called smooth tame estimates (see [Ha] and references there). We describe what they are and give a simple induction argument for proving them using a basic interpolation inequality. This approach works in a wide range of situations; we apply it to linear differential, pseudodifferential, and Fourier integral operators.

The description of the symbol calculus is included because, to obtain Moser-type estimates, we need exact finite order expansions and not the usual infinite asymptotic expansions that have infinitely smoothing error terms. The finite expansion is easily obtained by replacing the infinite Taylor series in the usual proofs with a finite Taylor series with an explicit error term. The ideas presented here are based upon [BGY, appendix to §V].

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have attempted to carry out the microlocal approach.

1. Statement of main theorem

Let X be a domain in $\mathbf{R}^n$. Given a function u: $X \to \mathbf{R}$, a **nonlinear differential operator of order p** is a smooth function $F(x, \partial^\alpha u(x), 0 \le |\alpha| \le p)$ of x, u(x), and the partial derivatives of u at x up to order p. We will denote $F(x, \partial^\alpha u(x))$ by F(u) for short. For convenience we have restricted our attention to differential operators acting on real scalar-valued functions. Nevertheless, everything done in this paper extends to real vector-valued functions and nonlinear differential operators acting on them, i.e. systems of PDE's. When necessary, we will indicate how this is done; also, see [D] and [Y1, Appendix] for a more general definition of a system of real principal type.

The **linearization** of a nonlinear differential operator F(u) at a function u is the linear differential operator F'(u) defined as follows:

$$F'(u)v = \left.\frac{d}{dt}\right|_{t=0} F(u+tv) .$$

Given a linear differential operator $P = \Sigma_\alpha a^\alpha(\partial/\partial x)^\alpha$ of order p, the **symbol of P** is defined to be the following function σ of the cotangent bundle of X, $T^*X \cong X \times \mathbf{R}^n$:

$$\sigma(x, \xi_i dx^i) = \Sigma_{|\alpha|=p}\, a^\alpha(x)\xi_\alpha , \tag{1.1}$$

where $\xi_\alpha = (\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}\dots(\xi_n)^{\alpha_n}$. If P is a matrix-valued differential operator, the symbol is defined to be the determinant of the righthand side. Also, to save trouble, we have made σ a real function; in order to make the defintion of σ agree with the notion of a symbol of a pseudodifferential operator, it is necessary to include a factor of $(\sqrt{-1})^p$.

Let σ be a symbol of a differential operator P of order p on X. Since σ is a function on

T^*X, which is a symplectic manifold, it induces a Hamiltonian vector field (defined in §B.2), denoted

$$H_\sigma = \frac{\partial\sigma}{\partial x^i}\frac{\partial}{\partial \xi_i} - \frac{\partial\sigma}{\partial \xi_i}\frac{\partial}{\partial x^i} \quad .$$

Note that σ is constant along the integral curves of H_σ which are called **bicharacteristics**. The integral curves of H_σ along which σ vanishes are called the **null bicharacteristics** of the operator P. They foliate the zero set of σ which is called the **characteristic variety**. A set $K \subset X$ is **pseudoconvex** with respect to P if any null bicharacteristic sitting over K leaves K, going forwards and backwards, in finite time. We say that it is **strongly pseudoconvex** with respect to P if *any* bicharacteristic curve sitting over K leaves K, going forwards and backwards, in finite time.

A key observation needed throughout this paper is that since the symbol $\sigma(x,\xi)$ and its Hamiltonian vector field H_σ are homogeneous in ξ, the construction of bicharacteristics and the solution of first order linear ODE's defined by H_σ can all be carried out on the bundle of unit covectors. If we restrict to a compact subset of X, the bundle is compact, allowing us to obtain uniform estimates on the length of the null bicharacteristics and on the norms of solutions to ODE's. Details can be found in the proof to Theorem(A.6.11).

We now specify which nonlinear PDE's are solved by our theorem. Recall that the differential operator P is **elliptic** if $\sigma(x,\xi)\neq 0$ whenever $\xi\neq 0$. Following [DH, Definition 6.3.2], the operator P is said to be **of real principal type at $x_0 \in X$** if:

(1.2) There exists a pseudoconvex neighborhood of x_0.

It is of **strong real principal type at $x_0 \in X$** if:

(1.3) There is a strongly pseudoconvex neighborhood of x_0.

:

Given a function u_0, a nonlinear differential operator F is of (strong) real principal type at u_0 and x_0 if $F'(u_0)$ is a linear differential operator of (strong) real principal type at x_0.

Remark. Using the fact that everything can be done on the bundle of unit covectors, which is compact, it is easy to see that (1.2) is equivalent to the following:

(1.2') There are no null bicharacteristics trapped over x_0.

Similarly, (1.3) is equivalent to

(1.3') There are no bicharacteristics trapped over x_0.

Using these observations, we obtain the following:

(1.4)Lemma. A linear differential operator P is of real principal type at x_0 if and only if it is of strong real principal type at x_0.

Proof. It suffices to show that if there is a bicharacteristic trapped over x_0, it must be a null bicharacteristic. This, however, follows easily from the homogeneity of the symbol. Along a trapped bicharacteristic, $\partial\sigma/\partial\xi = 0$ and if σ has order p, $\sigma = p^{-1}\xi\cdot\partial\sigma/\partial\xi = 0$.

Q.E.D.

The main result of this paper is the following:

Theorem 1. Let F be a nonlinear differential operator on $\overline{X}$. Let $f \in C^\infty(\overline{X})$ and $x_0 \in X$ be such that there exists $u_0 \in C^\infty(\overline{X})$ satisfying

$$F(x, \partial^\alpha u_0(x)) - f(x) = O(|x-x_0|^2),$$

and such that F is of real principal type at u_0 and x_0. Then there exists a function $u \in$

$C^\infty(\overline{X})$ such that $F(u) = f$ in a neighborhood of x_0.

By Lemma(1.4) it suffices to restrict our attention to differential operators of strong real principal type.

2. Applications

Before proving Theorem 1, we describe briefly some applications of Theorems 1 and 3. Sometimes we apply Theorem 3 (see §4) rather than Theorem 1 because the nonlinear PDE is not of real principal type as defined in §1, but solving the linearized equation can be reduced in a smooth tame way to solving a linear PDE of real principal type.

It is easily verified that the linear PDE

$$u_{xx} + a(x,y)u_{yy} = f,$$

is elliptic when $a > 0$, hyperbolic when $a < 0$, and of real principal type if whenever $a(x,y) = 0$, $\partial a/\partial x(x,y) \neq 0$. A nonlinear PDE whose linearization is of this form arises in the study of transonic flow. The sign of $a(x,y)$ reflects whether the flow is subsonic or supersonic.

A similar nonlinear PDE arises in the problem of isometrically embedding a given Riemannian 2-manifold with Gauss curvature changing sign cleanly (i.e. $dK(x) \neq 0$ if $K(x) = 0$) into $\mathbf{R}^3$. Our results therefore imply a theorem of C. S. Lin stating the local existence of such isometric embeddings (see [L]), although he is able to avoid using the Nash-Moser theorem and obtains much more precise results concerning the regularity of a solution.

Also, it is shown in [BGY, pp. 959-960] that the problem of isometrically embdding any Riemannian 3-manifold with nonvanishing curvature into $\mathbf{R}^6$ and the generic Riemannian 4-manifold into $\mathbf{R}^{10}$ reduces via the Nash-Moser implicit function theorem to solving a linear system of PDE's of real principal type and proving Moser-type estimates. For the 4-dimensional case they show that there exists a finite set of algebraic equations involving the curvature tensor and its covariant derivative such that if the equations do not all vanish, the linearized operator can be reduced to a differential operator of real principal type. We define a 4-dimensional Riemannian manifold to be **generic** if these algebraic equations never vanish when evaluated on the manifold. No explicit description of these algebraic conditions is given in [BGY]. Theorem 3 therefore implies the following:

(2.1)Theorem. Let M be a smooth n-dimensional Riemannian manifold with $n = 2, 3,$ or 4. Assume any of the following:

(a) $n = 2$ and the Gauss curvature K satisfies $dK(x) \neq 0$ if $K(x) = 0$.

(b) $n = 3$ and the Riemann curvature does not vanish.

(c) $n = 4$ and the Riemann curvature tensor is generic.

Then given $x \in M$, there exists a smooth isometric embedding of a neighborhood of x into $R^{n(n+1)/2}$.

Theorem 3 can also be applied to the local existence of metrics with a prescribed curvature tensor. Recall that the Riemann curvature of a smooth Riemannian metric g is a tensor of the form

$$\mathrm{Riem}(g) = (1/4)R_{ijkl}(dx^i \wedge dx^j)(dx^k \wedge dx^l), \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$

which depends smoothly on g and its derivatives up to second order. Now suppose that a tensor R that has the symmetries of the curvature tensor is given; is there a Riemannian metric whose curvature is R? This is a nonlinear second order system of PDE's; moreover, for $n > 3$ it is badly overdetermined (more equations than unknown functions). For $n = 3$, however, we get 6 equations for 6 functions. The equation is equivariant under the action of the group of diffeomorphisms; it is easily checked that this causes the system of PDE's to be degenerate. Nevertheless, a trick of D. DeTurck transforms the linearized system into a nondegenerate system. When his trick is applied to the question of prescribed Ricci curvature, the system to be solved is elliptic whenever the bilinear form defined by the Ricci tensor is nondegenerate. A surprising result is that the same does not happen for prescribed Riemann curvature in three dimensions, even though the Riemann curvature and the Ricci curvature are in some sense equivalent here. Nevertheless, assuming that the tensor R defines a nondegenerate bilinear form on Λ^2M, the linearized system can be reduced to a system of real principal type. Applying Theorem 3 and the Nash-Moser theorem, the local existence of metrics with prescribed (nondegenerate) curvature is obtained. We refer to [DY] for more details.

Another problem described in [DY] is the following: Let M be a smooth n-manifold. Fix a constant λ. Given a symmetric tensor $Q = Q_{ij}dx^i dx^j$, does there exist a Riemannian metric g on M such that

$$\mathrm{Ric}(g) + \lambda S(g)g = Q,$$

where Ric(g) is the Ricci curvature of g and S(g) the scalar curvature? For $\lambda \neq -1/2(n-1)$ and Q nondegenerate, DeTurck's trick reduces the problem to a nonlinear elliptic system and local solvability follows easily. On the other hand, when $\lambda = -1/2(n-1)$ and Q nondegenerate, DeTurck-Yang show that it reduces to a system of real principal type. Again, Theorem 3 applies here.

Finally, Theorem 3 can be used to solve nonlinear overdetermined systems of PDE's. In [Y1] it is shown that a certain class of linear overdetermined systems of PDE's--including elliptic ones--can be solved using the fundamental solution of a differential operator of real principal type. The corresponding nonlinear overdetermined system of PDE's can then be solved by applying the Nash-Moser theorem for exact sequences (see [Ha, §III.3]) and

Theorem 3. The precise statement is the following (see [Y1], [Y2] and [BCG] for definitions.):

(2.2)**Theorem.** Given a domain $X \subset \mathbf{R}^n$, let

$$F(x, u(x), \frac{\partial u}{\partial x}(x)) = 0 \tag{2.3}$$

be nonlinear first order system of PDE's for the vector valued function $u(x)$. Assume the following:

(a) The system of PDE's is involutive.

(b) There exists positive integers $m < n$ and s such that the reduced Cartan characters are $s_1 = \cdots s_m = s$, $s_{m+1} = \cdots s_n = 0$.

(c) There exists $x_0 \in X$ and a smooth function $u_0(x)$ such that

$$F(x_0,\ u_0(x_0), \frac{\partial u_0}{\partial x}(x_0)) = 0,$$

and the linear differential operator $F'(u_0)$ satisfies assumption (1.7) in [Y1].

Then there exists a smooth function $u(x)$ such that (2.3) holds in a neighborhood of x_0.

As indicated in the introduction, this is a generalization of the Cartan-Kähler theorem for a very special class of overdetermined systems of PDE's. For other results in this direction and a discussion of what the Cartan-Kähler theorem is, see [Y2].

3. Properties of linear differential operators of (strong) real principal type

In the course of proving Theorem 1, we need to know that if a differential operator is of (strong) real principal type, sufficiently small perturbations of it are also of (strong) real principal type. In particular, the following holds:

(3.1)Proposition. Let $P = \sum_{|\alpha|\leq m} a_\alpha(x)(\partial_x)^\alpha$ be a linear differential operator of (strong) real principal type at x_0. Then any differential operator $Q = \sum_{|\alpha|\leq m} b_\alpha(x)(\partial_x)^\alpha$ whose top order coefficients b_α, $|\alpha| = m$, are sufficiently close to a_α in the C^1 norm is also of (strong) real principal type at x_0. In particular, if $a_\alpha(x_0) = b_\alpha(x_0)$ and $da_\alpha(x_0) = db_\alpha(x_0)$, $|\alpha| = m$, then Q is of (strong) real principal type at x_0.

Proof. The proposition follows easily from the following facts:

(1)If the coefficients of P are C^1-close to those of Q, the corresponding coefficients of the Hamiltonian vector fields, when projected onto the bundle of unit covectors, are C^0-close.

(2) By the standard theorem on the continuous dependence of solutions of an ODE on a continuous parameter, the (null) bicharacteristics of Q are uniformly close to those of P (when projected onto the bundle of unit covectors).

Q.E.D.

Since there appears to be some confusion in the literature on the correct definition of real principal type, we want to discuss this briefly. Differential operators of real principal type have been studied primarily by microlocal analysts. An old definition of L. Hörmander was that P has "simple characteristics" if σ and $\partial\sigma/\partial\xi$ never vanish simultanenously. Using the equations defining the null bicharacteristics,

(3.2) $$x' = -\partial\sigma/\partial\xi, \quad \xi' = \partial\sigma/\partial x,$$

it is clear that this implies (1.2). Later, it was observed that the theorems and proofs on operators of real principal type in [DH] still hold microlocally under the following weaker assumption:

(3.3) For any (x,ξ), $\xi \neq 0$, such that $\sigma(x,\xi) = 0$, the exterior derivative of σ at (x,ξ), $d\sigma(x,\xi)$, is not a scalar multiple of the canonical 1-form $\xi_i dx^i$. This is equivalent to saying that H_σ is never a scalar multiple of the radial vector field

$$\xi_i \frac{\partial}{\partial \xi_i} .$$

For this reason (3.3) is the definition adopted by microlocal analysts and stated in most discussions on differential and pseudodifferential operators of real principal type. We shall say that an operator satisfying (3.3) is **microlocally of real principal type.** Unfortunately, it is not equivalent to (1.2). The simplest example of a differential operator that is microlocally but not locally of real principal type is

$$P = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{on} \quad \mathbb{R}^2$$

which has null bicharacteristics trapped over the origin and is not locally solvable there. A nondegenerate third order example can also be constructed without much difficulty.

4. Technical definitions and notations

Given a bounded open set $X \subset \mathbb{R}^n$ and $\overline{X}$ its closure, let $C^\infty{}_0(X)$ denote the space of smooth functions on $\mathbb{R}^n$ with support in $\overline{X}$ and $C^\infty(\overline{X})$ the space of smooth functions on X obtained by restricting smooth functions on $\mathbb{R}^n$. Given $f \in C^\infty(\overline{X})$, denote

$$\|f\|_k = \Big[\sum_{|\alpha| \le k} \int_X |\partial_x^\alpha f|^2 \, dx \Big]^{1/2} .$$

Let $H^k(\overline{X})$ be the completion of $C^\infty(\overline{X})$ with respect to the norm $\| \ \|_k$ and $H^k{}_0(X)$ the completion of $C^\infty{}_0(X)$ with respect to $\| \ \|_k$. Given k, $u_0 \in H^k$, and $\varepsilon > 0$, denote

$$B^k_\varepsilon(u_0) = \{u \in H^k \mid \|u - u_0\|_k < \varepsilon\}.$$

Throughout this paper we will fix three bounded open domains: $X \subset X' \subset X''$. Let

$$R\colon C^\infty(X'') \to C^\infty(\overline{X}) \text{ and } R'\colon C^\infty(X'') \to C^\infty(X')$$

denote restriction. We assuume that X and X' have smooth boundaries so that by [St] there exist linear extension operators

$$E\colon C^\infty(\overline{X}) \to C^\infty_0(X') \text{ and } E'\colon C^\infty(\overline{X}') \to C^\infty_0(X'')$$

which satisfy $RE = I$ and $R'E' = I$ and extend to bounded maps from H^k to H^k. We will define pseudodifferential operators and Fourier integral operators as maps from $C^\infty_0(X'')$ to $C^\infty_0(X'')$. Given such an operator $A\colon C^\infty_0(X'') \to C^\infty_0(X'')$, we use the restriction and extension operators to define a corresponding operator $RAE\colon C^\infty(\overline{X}) \to C^\infty(\overline{X})$. We will sometimes denote RAE simply by A.

More notation related to symbols, phase functions, and pseudodifferential and Fourier integral operators is defined in §§B.1-B.3.

5. Technical theorems

Theorem 1 follows rather easily from a semiglobal result (see §5 for the proof):

Theorem 2. Let F be a nonlinear differential operator on $\overline{X}$ where X is a bounded open set with smooth boundary and $u_0 \in C^\infty(\overline{X})$ such that the following hold:

(a) F is of strong real principal type at u_0.

(b) $\overline{X}$ is strongly pseudoconvex with respect to $F'(u_0)$.

(c) $F'(u_0)\colon C^\infty(\overline{X}) \to C^\infty(\overline{X})$ is a surjective map.

Then there exists a positive integer k and $\varepsilon > 0$ such that for any $f \in C^\infty(\overline{X}) \cap B^k{}_\varepsilon(F(u_0))$, there exists $u \in C^\infty(\overline{X})$ such that $F(u) = f$ on $\overline{X}$.

Theorem 2 in turn is proved by a direct application of the Nash-Moser implicit function theorem and the following result (see appendix A for a discussion of the Nash-Moser implicit function theorem; "smooth tame" is defined below and explained more thoroughly in appendix A.):

Theorem 3. Let F be a nonlinear differential operator on X, $u_0 \in C^\infty(\overline{X})$ satisfying the assumptions of Theorem 2. Then there exists j and δ such that given any $u \in C^\infty(\overline{X}) \cap B^j{}_\delta(u_0)$, the linear map $F'(u)$: $C^\infty(\overline{X}) \to C^\infty(\overline{X})$ is surjective and there exists a linear right inverse

$$E(u) : \ C^\infty(\overline{X}) \to C^\infty(\overline{X}), \quad F'(u)Q(u)f = f,$$

such that $E(u)f$ is a smooth tame map of u and f.

To say that $E(u)f$ is smooth tame means that it satisfies what we call "Moser-type estimates". In other words, given u_0, there exists j, δ, α, β such that for any $u \in C^\infty(K) \cap B^j{}_\delta(u_0)$ and $f \in C^\infty(K)$, the following estimate holds:

$$\|E(u)f\|_k \le C_k(\|f\|_{k+\alpha} + \|u\|_{k+\beta}\|f\|_\alpha), \quad k \ge j.$$

Here, the constant C_k depends on E, k, j, δ, but not on the functions u or f.

We provide a simple approach, one that does not use the full power of the modern theory of the Fourier integral operator. We use what we call "classical Fourier integral operators" which are globally defined integral operators with explicit kernels. Estimates for such operators are quite easy to prove. The key observation allowing us to use such operators is that given two symbols of strong real principal type that are near each other on a strongly pseudoconvex domain, there exists a global canonical transformation near the identity map which transforms one into the other. Using this we show that a differential operator of strong real principal type on a strongly pseudoconvex domain can be deformed to any nearby one by conjugating with globally defined classical Fourier integral operators. Therefore, no microlocalization is ever needed. The symbols of these operators are solved for using the symbol calculus for the composition of a Fourier integral operator with a pseudodifferential operator. The operator $E(u)$ is constructed by fixing a right inverse E_0 for the differential operator $F'(u_0)$ and using the (classical) Fourier integral operators that deform $F'(u_0)$ into $F'(u)$ to deform E_0 into a right inverse $E(u)$. Now, the dependence of E on u is easily seen by examining the classical Fourier integral operators, for which we have fairly explicit descriptions. We can then apply the Moser-type estimates proved in the appendix.

6. Proof of local solvability

The proof of Theorem 1 given Theorem 2 is well-known and straightforward. We quickly recall it here. First, use the Cauchy-Kovalevski theorem (see, for example, [J] or [F]) to extend the (p+1)th order Taylor expansion of u_0 centered at x_0 to a formal power series solution of $F(u) = f$. This requires that there exist a noncharacteristic covector $\xi \in T^*_{x_0}X$, but one always does for a differential operator of real principal type. By the Borel theorem (see[Ta, pp.40-41]) there exists a smooth function on X--call it u_0--such that its Taylor series at x_0 is the given formal power series. Since the old u_0 and the new u_0 agree at x_0 up to order $(p+1)$, the corresponding linearized operators agree up to first order. By

Proposition(3.1) the new linearized operator is still of real principal type on a neighborhood of x_0. Replace X by a bounded open neighborhood of x_0 which is pseudoconvex with respect to $F'(u_0)$ and for which $F'(u_0)$ is surjective; such a neighborhood exists by the local solvability theorem of a linear PDE of real principal type (see [DH] or [Y1, Appendix]).

Let ρ be a smooth compactly supported function on R^n which is identically 1 in a neighborhood of the origin. Given $\delta > 0$, let

$$f_\delta(x) = \rho(\delta^{-1}(x-x_0))f(x) + (1-\rho(\delta^{-1}(x-x_0))F(u_0)(x).$$

For δ sufficiently small, $f_\delta \in B^k{}_\varepsilon(F(u_0))$, where k and ε are as given in Theorem 2. By Theorem 2 there exists $u \in C^\infty(\overline{X})$ such that $F(u) = f_\delta$ on $\overline{X}$. Since $f_\delta = f$ in a neighborhood of x_0, Theorem 1 follows.

7. Deforming a symbol of strong real principal type into a nearby one

Here, we use the definitions and ideas given in §B.2, as well as notations and definitions given in §B.1. Let $\sigma_0(x,\xi)$ be a symbol of a differential operator of strong real principal type on $\overline{X}$ such that $\overline{X}$ is strongly pseudoconvex. Extend $\sigma_0(x,\xi)$ to be a compactly supported symbol on the larger bounded open set X'' and choose X', $X \subset X' \subset X''$ so that both X and X' are strongly pseudoconvex. We show that given any other symbol $\sigma(x,\xi)$ which is sufficiently close to σ_0 in the appropriate norm, there exists a canonical transformation $C: T^*X'' \to T^*X''$ near the identity map such that $\sigma = \sigma_0 \circ C$ on the compact set $\overline{X}'$. We will, in fact, solve for a phase function φ near $\varphi_0(x,\eta) = x^i\eta_i$ such that its associated canonical transformation C_φ deforms σ_0 into σ. In the next section we will use the phase function φ in constructing Fourier integral operators which will deform a differential operator with symbol σ_0 into a differential operator with symbol σ.

The precise result is the following:

(7.1)Proposition. Let $\sigma_0(x,\xi) \in \mathcal{S}_h{}^{p,\infty}$ be of strong real principal type on X'' such that X' is strongly pseudoconvex. Then there exists k, $\delta > 0$ such that given any homogeneous symbol σ of order p satisfying

$$\|\sigma - \sigma_0\|_{k,k} < \delta,$$

there exists a phase function $\varphi \in \mathcal{P}$ such that

(a) $\sigma(x,\xi) = \sigma_0 \circ C_\varphi(x,\xi)$ for $(x,\xi) \in T^*\overline{X}'$;

(b) $C_\varphi(X'' \times \mathbf{R}^n\backslash\{0\}) = X'' \times \mathbf{R}^n\backslash\{0\}$.

Moreover the map $\sigma \longmapsto \varphi$ can be defined so that it is smooth tame.

Proof. This is itself an application of the Nash-Moser implicit function theorem. We shall use the spaces and norms of symbols and phase functions defined in §§B1-B2.

Fixing the symbol σ_0, we want to invert the map

$$\Phi: \mathcal{P} \longrightarrow \mathcal{S}_h{}^{p,\infty}(\overline{X}'),$$

$$\varphi \longmapsto (\sigma_0 \circ C_\varphi)|_{\overline{X}'}$$

which is smooth tame. Recalling the definition of C_φ, $\sigma = \Phi(\varphi)$ is given implicitly as follows:

$$\sigma(x, \frac{\partial\varphi}{\partial x}) = \sigma\,(\frac{\partial\varphi}{\partial\eta}, \eta).$$

A straightforward computation then shows that the linearization of Φ at φ is given by $\sigma' = \Phi'(\varphi)\varphi'$, where $\Phi'(\varphi)$: $\mathcal{S}_h{}^{1,\infty}(X'') \longrightarrow \mathcal{S}_h{}^{p,\infty}(\overline{X}')$ and

$$\sigma'(x, \frac{\partial\varphi}{\partial x}) = \frac{\partial\sigma_0}{\partial x}(\frac{\partial\varphi}{\partial\eta},\eta)\frac{\partial\varphi'}{\partial\xi} - \frac{\partial\sigma}{\partial\xi}(x, \frac{\partial\varphi}{\partial x})\frac{\partial\varphi'}{\partial x} , \tag{7.2}$$

and $\sigma = \Phi(\varphi)$. Fix $\alpha > n/2$. If $\|\varphi - \varphi_0\|_{\alpha+1,1}$ and $\|\sigma - \sigma_0\|_{\alpha+2,2}$ are sufficiently small, the vector field acting on φ' in (7.2) is a homogeneous vector field which is "uniformly" close to the Hamiltonian vector field H_{σ_0} on X' in the C^1-norm. Therefore, since X' is strongly pseudoconvex, the integral curves of this vector field must pass through $\overline{X}'$ in finite time. Given $\sigma'(x,\xi)$, we can then apply theorem (A.6.11) to solve (7.2) on $\overline{X}'$ for φ'. Extending φ' to a compactly supported symbol of degree 1 on X'', we obtain a smooth tame inverse to $\Phi'(\varphi)$.

Using the Nash-Moser implicit function theorem (Theorem A.3.1), it then follows that given σ sufficiently close to σ_0 in the appropriate norm, there exists a phase function $\varphi \in \mathfrak{P}$ such that $\sigma = \sigma_0 \circ C_\varphi$ on $\overline{X}' \times R^n \backslash \{0\}$.

Q.E.D.

8. Deforming a differential operator of strong real principal type into a nearby one

Now let P_0 be a differential operator of strong real principal type on X'' for which X' $\subset$ X'' is strongly pseudoconvex. Given P sufficiently close to P_0 on X', let σ and σ_0 be the symbols of P and P_0, respectively, and let φ be the phase function which deforms σ_0 into σ as described in §7. We want to show that there exists an elliptic Fourier integral operator with phase function φ that deforms the operator P into P_0 modulo an operator of order less than p.

(8.1)Theorem. Let P_0 be a differential operator of strong real principal type on X'' such that X' is strongly pseudoconvex. Then given $N > 0$ and any differential operator P on X'' such that the symbol σ_0 of P_0 and the symbol σ of P satisfy the conditions of Theorem(7.1), there exists a symbol $b(x,\eta) \in \mathcal{S}^{0,\infty}$ such that $c^{\varphi}(x,D) = Pb^{\varphi}(x,D) - b^{\varphi}(x,D)P_0$ is a Fourier integral operator of order p-N when acting on functions compactly supported in X'. Moreover, for fixed N, the map from the coefficients of P to (φ,b,c) can be defined to be smooth tame.

Remarks.(1) For the proof of Theorem 3 we will not need anywhere near the full generality of this theorem. As we will see in §8, it will suffice to know Theorem(7.1) for $N = 1$. The proof of this case is trivial using the phase function φ given by Theorem(7.1)and the symbol calculus developed in §B.5. Nevertheless, since the proof for general N is not much more difficult, we include it here.

(2) It is possible to find a symbol $b(x,\eta)$ so that the error term is infinitely smoothing. This, however, requires using an infinite Taylor series in the symbol calculus and therefore an infinite number of derivatives of the coefficients of P. Such a construction could not be smooth tame.

Proof. The proof follows from the symbol calculus described in §B.5. We will be applying Theorem(A.6.1) which means we need to view everything as lying on a compact manifold with boundary. Again, this is possible if we assume that the appropriate symbols are homogeneous in the fiber variable ξ or η. Assume this to be true throughout the proof. In the end, the actual Fourier integral operator $b^{\varphi}(x,D)$ is defined by replacing $b(x,\eta)$ with the smooth symbol $b(x,\eta)(1 - \chi(\eta))$, where $\chi(\eta)$ is compactly supported and is identically 1 in a neighborhood of $\eta = 0$.

Let φ be the phase function given by Theorem(7.1). We will solve for $b(x,\eta) = b_0(x,\eta) + \cdots + b_N(x,\eta)$, where $b_k(x,\eta) \in \mathcal{S}_h^{-k,\infty}(X'')$, proceeding by induction on k. The idea is to compare terms which are symbols of the same order.

Let $P_0 = a_0(x,D) + \tilde{a}_0(x,D)$ and P be $a(x,D) + \tilde{a}(x,D)$, where $a_0(x,\xi) = \sigma_0(x,i\xi)$, $a(x,\xi)=\sigma(x,i\xi)$, and $\tilde{a}_0(x,\xi)$, $\tilde{a}(x,\xi)$ are symbols of order p-1. Now apply the symbol calculus derived in §B.5 to $[\, a(x,D) + \tilde{a}(x,D)\,]\, b^{\varphi}(x,D) - b^{\varphi}(x,D)[\, a_0(x,D) + \tilde{a}_0(x,D)]$. The object is to define $b_0,\ldots, b_N$ so that all the terms in the symbol calculus of order greater than or equal to -n vanish. First, observe that the top order term, which is of order p, is

$$a(x,\partial\varphi/\partial x)b_0(x,\eta) - b_0(x,\eta)a_0(\partial\varphi/\partial\eta,\, \eta) = (\sqrt{-1})^p b_0(x,\eta)(\, \sigma(x,\partial\varphi/\partial x) - \sigma_0(\partial\varphi/\partial\eta,\eta))$$
$$= 0,$$

by the definition of φ (see Theorem(7.1)). Observe that this already proves the theorem for N = 1.

Next, consider the terms of order p-1. A careful examination of the symbol calculus shows that we get an expression of the form:

(8.2) $\qquad H_\sigma b_0 + Ab_0\,,$

where $A(x,\eta)$ is a smooth symbol of order p-1 which depends smoothly on a, a_0, $\tilde{a}$, $\tilde{a}_0$, φ, and their derivatives up to second order. Applying Theorem(A.6.1), we can solve for $b_0(x,\eta)$, $(x,\eta) \in \overline{X}' \times \mathbf{R}^n\backslash\{0\}$ such that (8.2) vanishes. Using the extension operator E', extend b_0 to a smooth symbol on X''. Since we will want to invert the operator $b^{\varphi}(x,D)$, it is important that $b_0(x,\eta)$ be nonvanishing for $x \in X'$. This is easily attained by modifying the construction of the solution to the ODE described in Theorem(A.6.1); simply use sufficiently large positive initial data for each initial value problem on a co-ordinate chart. Note that in doing this we need a uniform bound on the length of the null bicharacteristics. This exists if $\|\sigma-\sigma_0\|_{\alpha+1,1} < C$ for some fixed constants $\alpha > n/2$, $C > 0$.

Finally, the each of the remaining terms of order p-k, $2 \le k \le N$, is of the form:

$$H_\sigma b_{k-1} + A_{k-1} b_{k-1} + B_{k-1}, \tag{8.3}$$

where A_{k-1} and B_{k-1} depend smoothly on a fixed finite number of derivatives of φ, $b_0, \ldots, b_{k-2}$, and the coefficients of P_0 and P. Applying Theorem(A.6.1), we solve for b_{k-1} on $\overline{X}' \times R^n \backslash \{0\}$ so that (8.3) vanishesand extend it to X''.

Since the symbol b is constructed using a sequence of smooth tame maps, it is smooth tame with respect to the coefficients of P. Also, using the error terms r_N and s_N in Theorem(B.5.2), the symbol $c(x,\eta)$ has an explicit (but extremely long) description in terms of a finite number of derivatives of φ, b, and the coefficients of P_0 and P. It therefore also is a smooth tame function of the coefficients of P.

Q.E.D.

9. Construction of a smooth tame right inverse

Let P_0 denote $F'(u_0)$ and let Q_0: $C^\infty(\overline{X}'') \to C^\infty(\overline{X}'')$ be a parametrix to P_0, as constructed in [DH]. We use Q_0 to construct a fundamental solution to P_0. Observe that $P_0Q_0 = I + S_0$, where S_0 is an infinitely smoothing operator. Let χ be a smooth compactly supported function on X'' that is identically 1 on X'. Then $PQ_0 - (1-\chi)S_0 = I + \chi S_0$, viewed as a map $C^\infty{}_0(X'') \to C^\infty{}_0(X'')$, is Fredholm of index 0. Assuming that P_0 is surjective, the operator Q_0 can then be modified on a finite dimensional subspace of functions so that it is still a parametrix and the operator $(I+\chi S_0)$: $C^\infty{}_0(X'') \to C^\infty{}_0(X'')$ is bijective. We then obtain a right inverse $E_0 = R'Q_0(I + \chi S_0)^{-1}E'$: $C^\infty(\overline{X}') \to C^\infty(\overline{X}')$ which extends to a continuous linear map

$$E_0\colon\ H^k(\overline{X}') \longrightarrow H^{k+p-1}(\overline{X}') \quad .$$

Now given u near u_0, the differential operator $P = F'(u)$ is close to P_0. Let $\varphi(x,\eta)$ be the phase function given by Theorem(7.1) and $B = b^\varphi(x,D)$, where $b(x,\eta)$ is identically 1. By Theorem(8.1), $PB - BP_0$ is an operator of order $p - 1$. Set $Q = RBE_0B^*E$ and observe that as operators on $C^\infty(\overline{X})$,

$$\begin{aligned} PQ &= RPBE_0B^*E \\ &= RBB^*E + R(PB - BP_0)E_0B^*E \\ &= I + S, \end{aligned}$$

where S is a operator of order 0. By restricting u so that P is sufficiently close to P_0 and φ is sufficiently close to φ_0 in the appropriate norm, the L^2-norm of S can be assumed to be less than 1/2. By Proposition(B.6.2) we obtain a right inverse for $P = F'(u)$,

$$E(u) = Q(I + S)^{-1} = BE_0B^*(I + S)^{-1} .$$

Moreover, by the theorems in §B.4 and §B.6, the operators B,C , and $(I + S)^{-1}$ are smooth tame functions of the coefficients of P which in turn depends smoothly on u and its partial derivatives up to order p (Strictly speaking, S is not a pseudodifferential operator but is constructed explicitly out of Fourier integral and pseudodifferential operators plus a single fixed operator E_0. This suffices for the proof of Moser-type estimates as carried out in Proposition(B.6.2)) . Therefore, the function E(u)f satisfies Moser-type estimates in terms of u and f.

Q.E.D.

Appendix A

A brief introduction to using the Nash-Moser implicit function theorem

A.1 Introduction

Although the Nash-Moser theorem can certainly be applied to other types of functionals, we shall focus on its use to solve a nonlinear PDE. A typical situation is the following: Let M be a domain in $\mathbf{R}^n$, and given $k \geq 0$, let H^k be the space of functions on M which are k-times differentiable in the L^2 sense (i.e., a Sobolev space). Let $H^\infty = \cap_k H^k$ be the space of smooth functions with finite norm in H^k for all $k \geq 0$. A nonlinear PDE of order p defines a smooth map

$$F: \ D^p \cap H^{k+p} \to H^k,$$

where D^p is a fixed open set in H^p. Now given $f \in H^k$, we want to solve for $u \in D^p$ satisfying

(A.1) $\qquad F(u) = f.$

Assume that we already have an approximate solution $u_0 \in D^p \cap H^{k+p}$ such that $F(u_0)$ is "close" to f. Can we perturb u_0 into a solution u to (A.1)? The standard implicit function theorem says that a solution $u \in D^p \cap H^{k+p}$ exists for all f sufficiently near $F(u_0)$ in H^k if the linearized operator at u_0,

$$F'(u_0): \ H^{k+p} \to H^k,$$

where $F'(u)v = d/dt|_{t=0} F(u+tv)$, has a bounded, linear right inverse E_0: $H^k \to H^{k+p}$ satisfying $F'(u_0)E_0 v = v$.

When F is a nonlinear differential operator, F'(u) is a linear differential operator of order p. The linear map F'(u) "loses p derivatives" and the standard implicit function theorem requires that we regain all p derivatives when we invert F'(u). Such an inverse exists for a linear differential operator P (= F'(u)) if and only if P satisfies a very restrictive assumption known as hypoellipticity. In particular, if P is of real principal type, a bounded right inverse exists if and only if it is elliptic. For the general differential operator of real principal type, the best we can do is a right inverse of the form

$$E:\ H^k \to H^{k+p-1},$$

one that "loses one derivative".

Nash confronted a similar difficulty in his proof of the existence of isometric embeddings. In fact, in his case, the right inverse is a zeroth order operator and loses all p derivatives. In this situation, given $f \in H^k$, there is no hope of solving for $u \in H^{k+p}$. Instead, let $f \in H^k$, where k is taken large, and we shall solve for $u \in H^j$, for some $j < k + p$.

One way of proving the standard implicit function theorem is to define a sequence u_0, $u_1, \ldots$ of better and better approximations to a solution by using an iteration scheme. Given u_0 and a right inverse E_0 to $F'(u_0)$, one usually uses Picard's scheme which is derived from the tangent line approximation to F at u_0:

$$u_{i+1} = u_i + E_0(f - F(u_i))\ .$$

If E_0: $H^k \to H^{k+p}$ is bounded and $|f-F(u_0)|_k$ is sufficiently small, the sequence $\{u_i\}$ will converge in H^{k+p}. Nash's idea was to modify this proof in the following way: First, use Newton's scheme:

$$u_{i+1} = u_i + E(u_i)(f - F(u_i)),\ \text{where } E(u) \text{ is the right inverse to } F'(u),$$

which, if it converges, converges at a much faster rate than Picard's scheme. Secondly, since there is no hope of this sequence converging anywhere if E(u) loses derivatives--each successive u_i will be less smooth than the previous one, Nash further modified the scheme by putting in a smoothing operator. Since Nash, in fact, used a very complicated scheme, we will describe Moser's simplification:

$$u_{i+1} = u_i + S_i E(u_i)(f-F(u_i)).$$

Here, S_i: $H^0 \to H^\infty$, $\lim_{i\to\infty} S_i = I$, helps to regain the derivatives lost by $E(u_i)$. This is now a delicate situation. On one hand, Newton's scheme drives u_i to a solution of (A.1) but loses derivatives. On the other, the smoothing operator regains the lost regularity but throws the sequence off the path towards the solution. Remarkably, if E(u) is well-behaved, Moser's scheme can be made to converge by defining the smoothing operators carefully and choosing the rate at which they converge to the identity operator appropriately. The catch is that the sequence does not converge in H^{k+p} but in H^j, where j is much smaller than k+p.

The crucial ingredients in the estimates that lead to convergence of Moser's scheme are interpolation inequalities which, given $i < j < k$, estimate $|u|_j$ in terms of $|u|_i$ and $|u|_k$ and the so-called Moser-type inequalities described in the next section.

As we have indicated before, Nash's original method was quite complicated but quite powerful in the sense that he obtains a solution in H^j where j is not much less than k+p. Of the many versions of the Nash-Moser theorem that exist in the literature, we believe that only [Hö3] attempts to reproduce Nash's optimal results (also, see [Gr]). Most follow Moser's ideas instead which are simpler but obtain a solution in H^j with j much smaller than k+p (so k must be taken to be large). Also, if $f \in H^\infty$, we want to obtain a smooth solution u $\in H^\infty$; it is shown in [Se] how to obtain this from Moser's scheme (also, see [Ha]).

A.2 Tame scales of Banach spaces

In describing the Nash-Moser implicit function theorem one has the choice of talking about a map between Fréchet spaces or between a scale of Banach spaces. Fréchet spaces are used by Sergeraert and Hamilton. They have the advantage of providing the natural abstract setting for the "C^∞" version of the theorem. We shall, however, use scales of Banach spaces which allows for a "C^k" as well as a "C^∞" theorem. Observe that the two terms are essentially the same since the topology of a Fréchet space is given by a set of seminorms and the scale of Banach spaces consists simply of the Banach spaces obtained by completing the Fréchet space with respect to each of the seminorms. The Fréchet space is simply the intersection of all the Banach spaces.

Definition. A **scale of Banach spaces** consists of a set of Banach spaces H^k, $k = 0, 1, \ldots$, with corresponding norms $|\ |_k$ which satisfy the following:

(A.2.1) $\qquad H^k \subset H^l\ ,\ \ k > l,$

(A.2.2) $\qquad |u|_l \le |u|_k\ ,\ u \in H^k,\ k > l.$

Let $H^\infty = \cap_k H^k$. The space H^∞ is a Fréchet space using the seminorms $|\ |_k$, $k \ge 0$.

The scale of Banach spaces is **tame** if there exists a 1-parameter family of linear smoothing operators

$$S_\theta\colon\ H^0 \to H^\infty,\ \ \theta \ge 1,$$

which satisfy the following estimates:

(A.2.3) $\qquad |S_\theta u|_k \le C_k \theta^{k-j} |u|_j\ ,\ \ u \in H^j,\ \ j \le k,$

(A.2.4) $\qquad |u - S_\theta u|_j \le C_k \theta^{j-k} |u|_k,\ \ u \in H^k,\ j \le k,$

(A.2.5) $\qquad \lim_{\theta\to\infty} |u - S_\theta u|_k = 0,\ \ u \in H^k.$

A tame scale of Banach spaces corresponds exactly to what Hamilton calls a tame Fréchet space; we leave the proof of this assertion as a straightforward exercise. In general, the index k indicates the order of differentiability of the functions in H^k. Examples include Hölder and Sobolev spaces. In particular, we will show in §A.4 that $H^k = H^{p,k}$ or $C^k(\mathbf{R}^n)$, define tame scales of Banach spaces.

If $H^k = C^k(M)$ or $L^p{}_k(M)$, $1 \le p \le \infty$, where M is a smooth compact manifold possibly with boundary, then one simply "cuts up" the function using a partition of unity subordinate to a collection of co-ordinate charts and smooths each piece individually using S_θ as defined above.

Definition. Let $\{E^k\}$ and $\{F^k\}$ be tame scales of Banach spaces and D^0 a bounded open subset of E^0. A map Φ: $D^0 \to F^0$ is **tame** if it satisfies the following:

(A.2.6) $\qquad \Phi(D^0 \cap E^k) \subset F^k$;

(A.2.7) $\qquad |\Phi(u)|_k \leq C_k |u|_k$.

We collect some obvious facts:

(A.2.8) <u>If</u> $\{H^k\}$ <u>is a tame scale of Banach spaces, so is</u> $\{H^{k+p}\}$.

(A.2.9) <u>Given tame scales</u> $\{E^k\}$ and $\{F^k\}$, <u>the scale</u> $\{E^k \times F^k\}$ <u>with norms</u>

$$|(u,v)|_k = |u|_k + |v|_k, \quad k \geq 0 \quad ,$$

<u>is also tame.</u>

(A.2.10) <u>Let</u> Φ: $D^0 \cap E^k \to F^k$ <u>and</u> Ψ: $\tilde{D}^0 \cap F^k \to G^k$, <u>where</u> $\tilde{D}^0$ <u>is an open set of</u> F^0, <u>be tame maps</u>. <u>Then the composition</u> $\Psi \circ \Phi$: $D^0 \cap \Phi^{-1}(\tilde{D}^0) \cap E^k \to G^k$ <u>is tame.</u>

As we shall see, the key to using the Nash-Moser theorem lies in proving that certain maps are tame. The definition of a tame map seems innocent enough, and one might wonder where the difficulty is. Usually the situation will be the following: Let $\{E^k\}$, $\{F^k\}$, and $\{H^k\}$ be tame scales. Then Φ will be a map of the following form:

$$\Phi: D^0 \cap (E^{k+\alpha} \times F^{k+\beta}) \to H^k,$$

where D^0 is an open (and usually bounded) subset of $E^\alpha \times F^\beta$. The map Φ is tame if an estimate of the form

$$|\Phi(u,v)|_k \leq C_k(|u|_{k+\alpha} + |v|_{k+\beta}), \quad (u,v) \in D^0 \cap (E^{k+\alpha} \times F^{k+\beta}).$$

This is what we call a Moser-type estimate. To see why this is usually not immediately obvious, let's look at a specific example. Let $E^k = F^k = H^k = C^k(\mathbf{R}^n)$ and $\Phi(u,v) = uv$. To say that Φ is tame means that the product of two functions satisfies an estimate of the form:

$$|uv|_k \le C_k(|u|_k + |v|_k) \,. \tag{A.2.11}$$

Although an estimate of the form $|uv|_k \le |u|_k|v|_k$ is easy, one like (A..2.11) is not. We will show in §A.4 that (A.2.11) holds as long as u and v are restricted to a fixed bounded subset of $C^0(\mathbf{R}^n)$. In other words, the constant C_k depends on $|u|_0$ and $|v|_0$.

More generally, u will determine a linear operator A(u) and the map Φ will be $\Phi(u,v) = A(u)v$. We will prove Moser-type estimates when A(u) is a linear differential operator, pseudodifferential operator, solution operator to a linear first order ordinary differential operator, and Fourier integral operator. The ultimate goal of this paper is to prove that Φ is tame when u determines the coefficients of a differential operator of real principal type and A(u) is a right inverse.

A.3 The Nash-Moser implicit function theorem

The following version of the Nash-Moser implicit function theorem is taken from [Sc] and [Se]:

(A.3.1)Theorem. Let $\{E^k\}$ and $\{F^k\}$ be tame scales of Banach spaces. Let D^k denote the unit ball in E^k and $\tilde{D}^k_\delta$ the ball of radius δ in F^k. Let $\Phi: D^0 \cap E^k \to F^k$ be a tame map satisfying the following:

$$\Phi: D^0 \cap E^k \to F^k \text{ is twice Fréchet differentiable.} \tag{A.3.2}$$

Let Φ' and Φ'' denote the first and second derivatives of Φ.

(A.3.3) $\Phi' : (D^0 \cap E^k) \times (D^0 \cap E^k) \to F^k$ is tame.

(A.3.4) $|(\Phi''(u)v,w)| \le C_k |u|_k |v|_k |w|_k, \quad u \in D^0 \cap E^k,\ v,\ w \in E^k.$

(A.3.5) There exists $\alpha > 0$ and a tame map

$$Q: (D^\alpha \cap E^{k+\alpha}) \times (\tilde{D}^k{}_1 \cap F^{k+\alpha}) \to E^k$$

such that $Q(u)v$ is linear in v and $\Phi'(u)Q(u)v = v$.

Then there exists $\delta > 0$, a positive integer β, and a tame map $\Psi: \tilde{D}^\beta{}_\delta \cap F^{k+\beta} \to D^\alpha \cap E^k$ such that $\Phi \circ \Psi(v) = v$. In particular, $\Psi(\tilde{D}^\beta{}_\delta \cap F^\infty) \subset D^\alpha \cap E^\infty$.

A.4 Smoothing operators on $\mathbf{R}^n$

Throughout this section, let $H^{p,k}$ denote the Banach space of functions whose derivatives up to order k are L^p-integrable and let

$$\|f\|_{p,k} = \left[\sum_{|\alpha| \le k} \int_X |\partial_x^\alpha f|^p \, dx \right]^{1/p}.$$

be the corresponding norm. For convenience we will sometimes denote $\|u\|_{p,0} = \|u\|_p$.

We shall recall from [Sc] how to construct a family of smoothing operators for functions on $\mathbf{R}^n$. The smoothing operator will be defined to be convolution with a given smooth "bump function" or mollifier. By choosing the bump function carefully and using Young's inequality, we will obtain (A.2.3-5).

Given functions u(x) and v(x), $x \in \mathbf{R}^n$, the **convolution** of u and v is

$$(u * v)(x) = \int_{\mathbf{R}^n} u(x-y)v(y)\, dy \quad .$$

We may assume that $u, v \in C^\infty{}_0(\mathbf{R}^n)$. The following properties of convolution play a crucial role in proving the necessary estimates:

(A.4.1) $$\frac{\partial}{\partial x^i}(u * v) = \frac{\partial u}{\partial x^i} * v = u * \frac{\partial v}{\partial x^i}$$

(A.4.2) $$\|u*v\|_p \le \|u\|_q \|v\|_r , \quad q^{-1} + r^{-1} - 1 = p^{-1}.$$

Estimate (A.4.2) is known as Young's inequality.

Define a smooth function a(x) as follows: Let $\hat{a}(\xi)$, $\xi \in \mathbf{R}^n$ be a smooth compactly supported function such that $\hat{a}(\xi) = 1$ for all ξ in a neighborhood of 0. Let

$$a(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\, \hat{a}(\xi)\, d\xi .$$

The function a(x) is a smooth Schwartz function, i.e. it satisfies the following estimates: given any multi-index α and positive integer N,

$$|(\partial_x)^{\alpha} a(x)| < C_{\alpha,N}(1+|x|)^{-N} .$$

Also, if we take the Fourier transform of a(x) and its derivatives and evaluate at $\xi = 0$, we find that

(A.4.3) $$\int a(x)\, dx = 1 \qquad \text{and} \qquad \int x^{\alpha}\, a(x)\, dx = 0 \ \text{ if } \ \alpha \neq 0.$$

The smoothing operators are then defined to be $S_\theta u = a_\theta * u$, $\theta \in \mathbf{R}$, where $a_\theta(x) = \theta^n a(\theta x)$. Observe that (A.4.1) implies that

(A.4.4) $$\partial_i S_\theta u = S_\theta \partial_i u .$$

We will prove that S_θ satisfies estimates (A.2.3) and (A.2.4), demonstrating that

Proposition. $\{H^{p,k}(\mathbf{R}^n)\}$ and $\{C^k(\mathbf{R}^n)\}$ are tame scales of Banach spaces.

Proof. First, observe that using (A.4.4), it suffices to prove the estimates for $0 = j \le k$.

Proving (A.2.3) is straightforward. Let β be a multi-index with $|\beta| \le k$ and consider

$$\|\partial^\beta S_\theta u\|_p = \|\partial^\beta a_\theta * u\|_p = \theta^{|\beta|} \|(\partial^\beta a)_\theta * u\|_p$$

$$= \theta^{|\beta|}\|(\partial^{\beta}a)_{\theta}\|_1\|u\|_p \text{ , by (A.4.2),}$$

$$= C\theta^{|\beta|}\|u\|_p \, .$$

Now sum over all β with $|\beta| \leq k$ to obtain the desired estimate.

Estimate (A.2.4) is a little trickier to prove. We will need the Nth order Taylor expansion of a smooth function:

(A.4.7)lemma. Let $f{:}[0,1] \to \mathbf{R}$ be a smooth function. Then

$$f(1) = \sum_{j=0}^{N} \frac{f^{(j)}(0)}{j!} + \frac{1}{N!}\int_0^1 (1-s)^N f^{(N+1)}(s)\, ds \, .$$

The following is a variant of Young's inequality:

(A.4.8)lemma. Given $u, v \in C^{\infty}{}_0(\mathbf{R}^n)$, $\theta \neq 0$, let

$$w(x) = \int_0^1 \int (1-s)^N v(y)u(x-s\theta^{-1}y)\, dy\, ds \, .$$

Then $\|w\|_p \leq (N+1)^{-1}\|v\|_1\|u\|_p$, $1 \leq p \leq \infty$.

proof. It suffices to show that given any $f \in C^{\infty}{}_0(\mathbf{R}^n)$,

$$\int f(x)w(x)\, dx \leq \|f\|_{p'}(N+1)^{-1}\|v\|_1\|u\|_p \quad ,$$

where $p^{-1} + p'^{-1} = 1$. Consider

$$\int f(x)w(x)\, dx = \int_0^1 \int \left(\int f(x)u(x-sy)\, dx \right) (1-s)^N v(y)\, dy\, ds \, .$$

Applying Hölder's inequality to the inner integral yields the desired estimate.

q.e.d.

Now consider

$$S_\theta u(x) = \int a_\theta(x-y)u(y)\,dy$$

$$= \int a(\theta(x-y))u(y)\,\theta^n dy = \int a(z)u(x-\theta^{-1}z)\,dz.$$

Let $f(t) = u(x-t\theta^{-1}z)$ and use (A.4.7) to obtain:

$$S_\theta u(x) = \sum_{j=0}^{k-1} \frac{(-\theta)^{-j}}{j!} \sum_{|\alpha|=j} \frac{\partial^\alpha u}{\partial x^\alpha}(x) \int z^\alpha a(z)\,dz$$

$$+ \frac{(-\theta)^{-k}}{(k-1)!} \sum_{|\alpha|=k} \int_0^1 \int (1-s)^k z^\alpha\; a(z)\frac{\partial^\alpha u}{\partial x^\alpha}(x-s\theta^{-1}z)\,dz\,ds\,.$$

Observe that all of the terms in the first summation vanish except when $\alpha = 0$. Applying lemma(A.4.8) to each term of the second summation proves (A.2.4).

Q.E.D.

Now let M be a compact n-dimensional manifold, possibly with boundary. As observed in §A.2, the use of co-ordinate charts, a corresponding partition of unity, and the extension operator defined in [St] implies the following:

Corollary. Given $1 \le p \le \infty$, $\{H^{p,k}(M)\,,\ k \ge 0\}$ is a tame scale of Banach spaces. Also, $\{C^k(M)\}$ is a tame scale.

A.5 An interpolation inequality and the basic Moser-type estimates

The following interpolation inequality plays a crucial role in proving Moser-type estimates:

(A.5.1)Lemma. Given a tame scale of Banach spaces $\{H^k\}$, $i < j < k$,

$$|u|_j < C_k |u|_i^{(k-j)/(k-i)} |u|_k^{(j-i)/(k-i)}, \quad u \in H^k.$$

Proof.

$$|u|_j \le |u - S_\theta u|_j + |S_\theta u|_j$$
$$\le C_k \theta^{j-k} |u|_k + C_j \theta^{j-i} |u|_i.$$

Now set $\theta = (|u|_k / |u|_i)^{1/(k-i)} \ge 1.$

q.e.d.

The following corollary will produce the Moser-type estimates that we need:

(A.5.2)Corollary. Let $\{E^k\}$ and $\{F^k\}$ be tame scales of Banach spaces. Given $a<b, c<d$ such that $a + d = b + c$,

$$|u|_b |v|_c \le C_d (|u|_a |v|_d + |u|_d |v|_a) \ , \ \text{for any } u \in E^d \text{ and } v \in F^d.$$

Proof. First, observe that given $0 < \alpha < 1$ and $x, y > 0$, $x^\alpha y^{1-\alpha} \le x + y$. Now, applying the proposition,

$$|u|_b |v|_c \le C(|u|_a)^{(d-b)/(d-a)} (|u|_d)^{(b-a)/(d-a)} (|v|_a)^{(d-c)/(d-a)} (|v|_d)^{(c-a)/(d-a)} .$$

Since $d - b = c - a$ and $b - a = d - c$,

$$|u|_b |v|_c \le C(|u|_a |v|_d)^{(c-a)/(d-a)} (|u|_d |v|_a)^{(d-c)/(d-a)}$$
$$\le C(|u|_a |v|_d + |u|_d |v|_a) .$$

The constant C depends on a, b, c, and d but since all are nonnegative, it can be taken to depend only on d.

q.e.d.

From now on, we shall use Sobolev spaces. Let M be a compact manifold possibly with boundary. Let $H^{p,k}$ denote the space of (sometimes $\mathbf{R}^m$ valued-)functions on M whose derivatives up to order k are bounded in L^p, $1 \le p \le \infty$. As before, denote the norm on $H^{p,k}$ by $\|u\|_{p,k}$. Everything in this section also applies to the scale $\{C^k(M)\}$. Analogous results also hold for $\{C^{k,\alpha}(M)\}$, $0 < \alpha < 1$; see [Hö3]. Later, when we turn to estimating pseudodifferential and Fourier integral operators, we will use only $p = 2$.

First, recall the Sobolev inequality(see [Au] or [St] for proofs.):

Lemma. Let M be a compact Riemannian manifold of dimension n, possibly with boundary. If $r^{-1} > q^{-1} - k/n$, then $H^{q,k}(M)$ is continuously embedded in $H^r(M)$. In particular, if $q > n/k$, there exists a constant C_q such that

$$\|u\|_{\infty,0} \le C_q\|u\|_{q,k}\ , \quad \text{for all } u \in H^{q,k}.$$

The fundamental Moser-type estimate is for the product of two functions. We give a proof by induction, using the interpolation inequality above.

(A.5.3)Lemma. Given smooth functions f and g on M, $1 \le p \le \infty$, $k \ge 0$, and $\upsilon > n/p$, the following holds:

$$\|fg\|_{p,k} \le C_k(\|f\|_{p,k}\|g\|_{p,\upsilon} + \|f\|_{p,\upsilon}\|g\|_{p,k})\ ,$$

where C_k depends on M, p, k, υ but not on f and g.

Proof. By the Hölder and Sobolev inequalities,

$$\|fg\|_{p,0} \le \|f\|_{p,0}\|g\|_{\infty,0} \le C\|f\|_{p,0}\|g\|_{p,\upsilon}.$$

Now assume that the inequality holds for $k = j \ge 0$ and consider $k = j + 1$. It suffices to look at

$$\|\partial_i(fg)\|_{p,j} \le \|(\partial_i f)g\|_{p,j} + \|f(\partial_i g)\|_{p,j}$$

$$\le C_j(\|\partial_i f\|_{p,j}\|g\|_{p,\upsilon} + \|\partial_i f\|_{p,\upsilon}\|g\|_{p,j} + \|f\|_{p,j}\|\partial_i g\|_{p,\upsilon} + \|f\|_{p,\upsilon}\|\partial_i g\|_{p,j}),$$

by induction assumption,

$$\le C_j(\|f\|_{p,j+1}\|g\|_{p,\upsilon} + \|f\|_{p,\upsilon+1}\|g\|_{p,j} + \|f\|_{p,j}\|g\|_{p,\upsilon+1}+ \|f\|_{p,\upsilon}\|g\|_{p,j+1}).$$

Now apply Corollary(A.5.2) with $a = \upsilon$ and $d = j+1$ to obtain the desired estimate.

Q.E.D.

Later, we will use the same basic induction argument to prove Moser-type esimates for classical pseudodifferential and Fourier integral operators.

Recall one particular case of the Gagliardo-Nirenberg inequality (see [Ga], [Ni3], or [Au,pp.93-96] for proofs):

Lemma. $\|f\|_{q,\alpha} \le C_{p,q,\alpha}(\|f\|_{\infty,0})^{1-p/q}(\|f\|_{p,\beta})^{p/q}, \quad p \le q, \ p\beta = q\alpha.$

Next, we consider the composition of smooth functions.

(A.5.4)Lemma. Let B be a bounded open set in $\mathbf{R}^m$, $\upsilon > n/p$, and

$$D = \{u \in H^{p,\upsilon} \mid u(M) \subset B\}.$$

Then given $k \ge 1$, any smooth function φ: $B \to \mathbf{R}^1$, and $u \in D$,

$$\|\varphi\circ u\|_{p,k} \le C(1 + \|u\|_{\infty,1})^{k-1}(\|\varphi\|_{\infty,1}\|u\|_{p,k} + \|\varphi\|_{\infty,k}\|u\|_{p,1}) .$$

In particular, the map $(\varphi, u) \longmapsto \varphi\circ u$ is smooth tame.

Proof. Observe that when M has dimension 1, the following formula holds for $k \ge 1$:

$$\frac{\partial^k(\varphi\circ u)}{\partial x^k} = \sum_{m=1}^{k} \sum_{\substack{\alpha_1+\cdots\alpha_m=k \\ \alpha_j\geq 1}} C_{m,\alpha} \frac{\partial^m\varphi}{\partial u^m} \circ u \frac{\partial^{\alpha_1}u}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}u}{\partial x^{\alpha_m}} .$$

When $n > 1$, there is an analogous but more complicated formula for the partial derivatives of $\varphi\circ u$ of order k. In any case, applying Hölder's inequality, we get

$$\|\partial_x^k(\varphi\circ u)\|_{p,0} \leq C\sum_{m,\beta} \|\varphi\|_{\infty,m}\|\partial u\|_{q_1,\beta_1} \cdots \|\partial u\|_{q_m,\beta_m},$$

where $\beta = (\beta_1,\ldots, \beta_m)$, $\beta_j = \alpha_j - 1$, $q_j = p(k-m)/\beta_j$, $1 \leq j \leq m$, and ∂u is the gradient of u. Applying the Gagliardo-Nirenberg inequality,

$$\|\partial u\|_{q_j,\beta_j} \leq C(\|\partial u\|_{\infty,0})^{1-\beta_j/k}(\|\partial u\|_{p,k-m})^{\beta_j/k} ,$$

and therefore,

$$\|\partial_x^k(\varphi\circ u)\|_{p,0} \leq C\sum_m \|\partial u\|_{\infty,0}^{m-1} \|\varphi\|_{\infty,m}\|\partial u\|_{p,k-m} .$$

The lemma now follows easily by applying corollary(A.5.2) with $a = 1$, $b = m$, $c = k - m$, and $d = k - 1$ and by summing the estimate over all derivatives of $\varphi\circ u$ of order k or less.

Q.E.D.

The following lemmas which give Moser-type estimates for linear and nonlinear differential operators follow easily from the lemmas above:

(A.5.5)Lemma. Let $\upsilon > n/p$. The map

$$\Phi\colon D \cap H^{p,k+\upsilon}\times\cdots\times H^{p,k+\upsilon}\times H^{p,k+\mu+\upsilon} \to H^{p,k+\upsilon}$$

$$(a^\alpha(x),\ |\alpha| \leq \mu;\ u(x) \longmapsto \Sigma_\alpha a^\alpha(x)\partial_\alpha u(x) ,$$

where D is a bounded subset of $H^{p,\upsilon}\times\cdots\times H^{p,\upsilon}\times H^{q,\mu+\upsilon}$, is a tame map.

(A.5.6)Lemma. Let $\upsilon > n/p$ and D a bounded subset of $H^{\upsilon+\mu}$. A smooth nonlinear differential operator of order μ,

$$F:\ D \cap H^{k+\upsilon+\mu} \to H^{k+\upsilon},$$

is a tame map.

A.6 Moser-type estimates for global solutions to linear first order ODE's

For this section let $H^k = H^{2,k}$ and $\|u\|_k = \|u\|_{2,k}$. We want to prove the following:

(A.6.1)Theorem. Let X be a smooth compact manifold with boundary. Let V_0 be a smooth nonvanishing vector field on X such that every integral curve has finite length and both endpoints lying on the boundary of X.

There exists $\delta > 0$ such that for any smooth vector field V on X satisfying

$$\|V - V_0\|_{\infty,1} < \delta,$$

and smooth function b on X, there exists a linear map

$$\Phi(V,b):\ H^0 \to H^0,$$

such that for any $f \in C^\infty(X)$,

$$(V + b)\Phi(V)f = f, \tag{A.6.2}$$

$$\|\Phi(V,b)f\|_k \le C_k[\ (1+\|b\|_{\infty,1})\|f\|_k + (\|V\|_{\infty,k+1}+\|b\|_{\infty,k+1})\|f\|_n\ ],\quad k \ge 0. \tag{A.6.3}$$

The constant C_k depends only on δ, V_0, X and k. In particular, Φ is a smooth tame function of V, b, and f.

Proof. First, embed X smoothly into a larger open n-manifold X' and extend V_0 to a smooth nonvanishing vector field on X'. Since X is compact, there is a finite set of (n-1)-dimensional balls $B_1, \ldots, B_N$ smoothly embedded in X' satisfying the following:

(A.6.4) V_0 <u>is never tangent to</u> B_α, $1 \le \alpha \le N$.

(A.6.5) X <u>is covered by</u> $U_1, \dots, U_N$, <u>where</u> U_α <u>is the open set containing all the integral curves of</u> V_0 <u>intersecting</u> B_α.

Under these assumptions, the sets $U_1, \dots, U_N$ form an atlas of co-ordinate charts for X with the following co-ordinates:

$(x_\alpha^1, \dots, x_\alpha^n) \longmapsto$ the point in U_α reached by flowing for time x_α^n along the integral curve starting from $(x_\alpha^1, \dots, x_\alpha^{n-1}) \in B_\alpha$.

We can then define the Sobolev norm $\| \ \|_k$ with respect to these co-ordinates, so that

$$\|u\|_k^2 = \Sigma_\alpha \Sigma_{|\beta| \le k} \int |\partial^\beta u|^2 \, dx_\alpha,$$

where the partial derivatives are taken with respect to the co-ordinates $x_\alpha^1, \dots, x_\alpha^n$, and for a vector field V,

$$\|V\|_k^2 = \Sigma_\alpha \Sigma_{i \le n} \|v_\alpha^i\|_k^2 ,$$

where on U_α the vector field $V = v_\alpha^i(\partial/\partial x_\alpha^i)$. Any other way of defining the norms gives equivalent norms.

Given a smooth vector field V on X, we first use the extension operator E described in §1 to extend it to X'.

For δ sufficiently small and $\|V-V_0\|_{\infty,0} < \delta$, the vector field V will be nonvanishing in X and never tangent to $B_1, \dots, B_N$. Each of its integral curves will also still pass through X in finite time and lie entirely within at least one of the $U_1, \dots, U_N$.

On each co-ordinate chart U_α,

$$V_0 = \frac{\partial}{\partial x^n_\alpha} \qquad \text{and} \qquad V = v^i \frac{\partial}{\partial x^i_\alpha} \quad ,$$

where $v^1, \ldots, v^{n-1}$ are uniformly close to 0 and v^n is close to 1.

Let $h_1, \ldots, h_M$ be a set of compactly supported smooth functions on X' such that

(A.6.8) $\qquad \operatorname{supp} h_\mu \subset U_\alpha$ for some α ,

(A.6.9) $\qquad \Sigma_\mu h_\mu = 1$ on X.

We now define $\Phi(V)$ as follows: Given $f \in C^\infty(X)$, extend it to X' using the extension operator E. For each $\mu = 1, \ldots, M$, suppose that $\operatorname{supp} h_\mu \subset U_\alpha$. Now let u_μ be the unique smooth solution on U_α to

(A.6.10) $\qquad (V + b)u_\mu = h_\mu f, \quad u_\mu\big|_{B_\alpha} = 0$.

Since the integral curves of V pass through X and do not become "trapped", u_μ is a well-defined function on U_α. Also, u_μ extends smoothly to be zero on $X \backslash U_\alpha$. Define

$$\Phi(V,b)f = \Sigma_\mu u_\mu \quad .$$

$\Phi(V,b)f$ clearly satisfies (A.6.2).

In co-ordinates, equation (A.6.10) looks like the following:

$$v^n \frac{\partial u}{\partial x^n} + \sum_{i=1}^{n-1} v^i \frac{\partial u}{\partial x^i} + bu = h_\mu f \ , \quad u\Big|_{x^n = 0} = 0 \ .$$

This is a 1-by-1 symmetric hyperbolic system, where x^n should be viewed as a time co-ordinate. Moser-type estimates, i.e. (A.6.3), for solutions to a symmetric hyperbolic system on a bounded domain are proved in [K], [BGY], and [Ha].

Q.E.D.

Moser-type estimates can also be obtained for solutions to ODE's on the cotangent bundle of a compact manifold with boundary, as long as everything is homogeneous in the fiber direction. We shall study linear first order ODE's on $T^*X\backslash\{0\}$ involving homogeneous symbols as defined in §B.2. We shall use the homogeneous norms $\| \|_{j,k}$ as defined in §B.1. Given an integer m, a vector field V on $T^*X\backslash\{0\}$ is **homogeneous of order m** if

$$[\xi_i\frac{\partial}{\partial\xi_i}, V] = -mV.$$

Observe that as a linear differential operator, V: $\mathcal{S}_h{}^{p,\infty} \longrightarrow \mathcal{S}_h{}^{p+m,\infty}$. It is also easily checked that when V is written in terms of $\partial/\partial x^i$ and $\partial/\partial\xi_i$, its coefficients are homogeneous symbols. By $\|V\|_{j,k}$ we shall mean the sum of the corresponding norms of the coefficients. We then have the following modified version of Theorem(A.6.1):

(A.6.11)Theorem. Let V_0 be a smooth nonvanishing homogenenous vector field of order k on $T^*M\backslash\{0\}$ such that every integral curve has finite length and both endpoints lying over the boundary of M.

Fix $\alpha > n/2$. There exists $\delta > 0$ such that for any smooth homogeneous vector field V of degree m on T^*M satisfying

$$\|V - V_0\|_{\alpha+1,1} < \delta,$$

and symbol $b \in \mathcal{S}_h{}^{m,\infty}$, there exists for each integer p, a linear map

$$\Phi(V,b)\colon\ \mathcal{S}_h{}^{p+m,0,0} \longrightarrow \mathcal{S}_h{}^{p,0,0}$$

such that for any $f \in \mathcal{S}_h{}^{p+m,\infty}$,

(A.6.12) $$(V+b)\Phi(V,b)f = f,$$

(A.6.13) $$\|\Phi(V,b)f\|_{\alpha+k,k} \le C_k[\,(1+\|b\|_{\alpha,0})\|f\|_{\alpha+k,k} + (\|V\|_{k+\alpha,k}+\|b\|_{k+\alpha,k})\|f\|_{\alpha,0}],$$

$k > n/2.$

$$(s, u) \longmapsto v \ (= (I - s(x,D))^{-1}u),$$

such that $(I - s(x,D))v = u$.

Proof. Choose D such that the L^2-operator norms of both $s(x,D)$ and $s(x,D)^*$: $C^\infty(\overline{X}') \to C^\infty(\overline{X}')$ are less than 1/2. The operator must then be bijective , so that the desired map is simply $(s,u) \longmapsto (I - s(x,D))^{-1}u$.

Fix $\alpha \geq n + 1$ and choose D so that there exists $R > 0$ such that $\|s\|_{\alpha,0} \leq R$ for all $s \in D$.

The Moser-type estimates

(B.6.3) $\qquad \|v\|_k \leq C_k(\|s\|_{\alpha+k,0}\|u\|_1 + \|u\|_k), \ \ s \in D, \ u \in H^k, \ v = (I-s(x,D))^{-1}u,$

are proved by induction as follows: Denote $S = s(x,D)$ and observe that if $v = (I - S)^{-1}u$,

$$v = u + Sv.$$

Therefore, $\|v\|_0 \leq 2\|u\|_0$. Now consider

(B.6.4) $\qquad \|v\|_k \leq \|u\|_k + \|Sv\|_k$.

We shall omit the details, but a slight modification of the proof to Theorem(B.4.1) with $\varphi = \varphi_0$ gives the following estimate:

(B.6.5) $\qquad \|Sv\|_k \leq \|S\|_0\|v\|_k + C_k(\|s\|_{\alpha+k,0}\|v\|_1 + \|s\|_{\alpha,0}\|v\|_{k-1}),$

where $\|S\|_0$ is the L^2 operator norm of S. The estimate (B.6.3) then follows by substituting (B.6.5) into (B.6.4) and applying the inductive assumption.

Q.E.D.

It then follows from the results proved here that given an elliptic Fourier integral operator $F = a^\varphi(x,D)$ near the identity operator, there exists a right inverse

$$F^{-1} = a^\varphi(x,D)^* b(x,D)(I-S)^{-1},$$

where $b(x,D)$ is the parametrix to $a^\varphi(x,D)a^\varphi(x,D)^*$ and $S = I - a^\varphi(x,D)a^\varphi(x,D)^* b(x,D)$.

Moreover, the map $(\varphi,a,u) \longmapsto F^{-1}u$ is smooth tame for all phase functions φ near φ_0, symbols a near any fixed elliptic symbol a_0, and $u \in C^{\infty}(\overline{X}')$.

The constant C_k depends only on δ, V_0, X, p, m, α, and k. In particular, Φ is a smooth tame function of V, b, and f.

Proof. The trick is to reduce the homogeneous ODE to an equation on the bundle of unit covectors, which is a compact manifold with boundary, and apply Theorem(A.6.1).

Fixing any smooth Riemannian metric on X, let S^*X denote the bundle of unit covectors. Observe that a natural homogeneous Riemannian metric is also induced on T^*X. For each p, there is an obvious isomorphism $\mathcal{S}_h^{p,\infty} \cong C^\infty(S^*X)$. The maps back and forth are clearly smooth tame. Now given any homogeneous vector field V of order m on T^*X, there is a unique splitting $V = V' + \upsilon(\xi_i\partial/\partial\xi_i)$ such that V' is orthogonal to $\xi_i\partial/\partial\xi_i$. In particular, V' along S^*X is the orthogonal projection of V onto the tangent space of S^*X. Now, given homogeneous symbols b of order m and u of order p,

$$(V+b)u = (V' + \upsilon(\xi_i\partial/\partial\xi_i) + b)u = (V' + p\upsilon + b)u. \tag{A.6.14}$$

Observe that the last expression is well-defined when restricted to S^*X. In fact, given a homogeneous symbol f of order p+m, solving $(V+b)u = f$ on T^*X is equivalent to solving $(V' + p\upsilon + b)u = f$ on S^*X. The theorem now follows directly from Theorem(A.6.1), if we can show that the integral curves of V' have finite length and both endpoints on the boundary of S^*X. The details are left to the reader, but one simply verifies that if $(x(t),\xi(t))$ is a an integral curve of V' then there exists a smooth function $\rho(t)$ such that $(x(t), \rho(t)\xi(t))$ is an integral curve of V.

Q.E.D.

Appendix B

Classical pseudodifferential and Fourier integral operators

We present here a stripped down theory of pseudodifferential and Fourier integral operators. By sticking to a special class of operators on R^n which can be described and estimated explicitly, the necessary estimates--some well-known and some new--and the symbol calculus are easily derived. The price paid is that many of the ideas and motivation underlying the general theory are lost in our simplified proofs. For these we recommend [BFG], [GS], [Hö2], [Ni2], and [Tr].

B.1 Symbols and pseudodifferential operators

For convenience, everything will be compactly supported and smooth on a bounded open set $X'' \subset R^n$. Throughout this appendix, $H^k = H^{2,k}$ and $\|u\|_k$ denotes $\|u\|_{2,k}$.

Given $u \in C^\infty{}_0(R^n)$, denote the **Fourier transform** of u by

(B.1.1) $$\hat{u}(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\xi}\, u(x)\, dx \ ;$$

the corresponding inverse transform is

$$u(x) = \int e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi \ .$$

Recall the Plancherel theorem which states that $\|\hat{u}\|_0 = (2\pi)^{n/2}\|u\|_0$. Observe that if the support of u lies in a bounded set X'', then applying Hölder's inequality to (B.1.1) yields

(B.1.2) $$|\hat{u}(\xi)| \le (2\pi)^{-n}[\mathrm{vol}(X'')]^{1/2}\|u\|_2 \ .$$

A **symbol of order p** is a smooth function $a(x,\xi)$, $(x,\xi) \in R^n \times R^n$ which satisfies the following estimates: given multi-indices α,β, there exist a constant $C_{\alpha,\beta}$ such that

$$|(\partial_x)^{\alpha}(\partial_{\xi})^{\beta}a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{p-|\beta|} .$$

Throughout this paper we restrict to symbols $a(x,\xi)$ such that for any $\xi \in R^n$, $a(\cdot,\xi)$ is compactly supported in X''. Let $\mathcal{S}^{p,\infty}$ denote the space of all such symbols of order p. We will use two different kinds of norms for symbols. First, let

$$\|a\|_{j,k} = \sup_{\xi} (1+|\xi|)^{-p+k} \Sigma_{|\beta|\le k}\|(\partial_{\xi})^{\beta}a(\cdot,\xi)\|_{2,j} ,$$

where by $\|a(\cdot,\xi)\|_{2,j}$ we mean the $H^{2,j}$-norm of $a(\cdot,\xi)$ as a function of x and with ξ fixed.

Let $\mathcal{S}^{p,j,k}$ be the completion of $\mathcal{S}^{p,\infty}$ with respect to the norm $\| \|_{j,k}$. Observe that for fixed p and j, $\{\mathcal{S}^{p,j,k}\}$ forms a tame scale of Banach spaces.

When using the symbol calculus, we will need to use homogeneous symbols. A function $a(x,\xi)$, $(x,\xi) \in X'' \times R^n\backslash\{0\}$ is a **homogeneous symbol of order p** if given any $t > 0$, $a(x,t\xi) = t^p a(x,\xi)$. As before, we always assume that a homogeneous symbol $a(x,\xi)$ is compactly supported in X'' as a function of x. Let $\mathcal{S}_h{}^{p,\infty}$ denote the space of homogeneous symbols of order p. For homogeneous symbols, the symbol norms will be defined as follows:

$$\|a\|_{j,k} = \sup_{\xi} |\xi|^{-p+k} \Sigma_{|\beta|\le k}\|(\partial_{\xi})^{\beta}a(\cdot,\xi)\|_{2,j} .$$

Let $\mathcal{S}_h{}^{p,j,k}$ denote the completion with respect to the norm $\| \|_{j,k}$. Throughout this paper, we will fix a smooth compactly supported function $\chi(\xi)$ such that $\chi = 1$ in a neighborhood of $\xi = 0$ and use it to define a embedding

$$\mathcal{S}_h{}^{p,j,k} \longrightarrow \mathcal{S}^{p,j,k}$$

$$a(x,\xi) \longmapsto (1-\chi(\xi))a(x,\xi) .$$

Next, denote

$$\hat{a}(\eta,\xi) = (2\pi)^{-n} \int e^{-ix\cdot\eta}\, a(x,\xi)\, dx .$$

Define another set of norms on $\mathcal{S}^{p,\infty}$ as follows:

$$|a|_{j,k} = \sup_{(\eta,\xi)} (1+|\eta|)^{-j}(1+|\xi|)^{-p+|\beta|} |(\partial_{\xi})^{\beta}\hat{a}(\eta,\xi)| .$$

Since the support of $a(\cdot,\xi)$ always lies inside a fixed bounded set X'', estimate (B.1.2) implies that $|a|_{j,k} \le C\|a\|_{j,k}$.

Given a symbol $a(x,\xi)$ of order p, we will denote the corresponding **(classical) pseudodifferential operator** by

$$a(x,D)u(x) = \int e^{ix\cdot\xi}\, a(x,\xi)\hat{u}(\xi)\, d\xi .$$

Observe that given a differential operator $P = a^{\alpha}(\partial_x)^{\alpha}$, we have $P = a(x,D)$, where $a(x,\xi) = a^{\alpha}(x)(i\xi)^{\alpha}$.

Since pseudodifferential operators are defined using the Fourier transform, they are most naturally studied in the setting of $H^{2,k}$ spaces. Recall that a differential operator of order p is a bounded operator as map of $H^{2,k+p}$ to $H^{2,k}$. The same is true of a pseudodifferential operator of order p. Since we will need Moser-type estimates later, we also need to know how the operator norm of a(x,D) depends on the symbol a. In fact, when estimating the operator norm of a Fourier integral operator, we will need an estimate for a more general type of pseudodifferential operator. A smooth function $a(x,y,\xi)$, $(x,y,\xi) \in R^n \times R^n \times R^n$, is also called a **symbol of order p** if it satisfies the following estimates:

$$|(\partial_x)^{\alpha}(\partial_y)^{\beta}(\partial_{\xi})^{\gamma}a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1 + |\xi|)^{p-|\gamma|} .$$

As always, we assume that for fixed ξ, $a(\cdot,\cdot,\xi)$ is compactly supported in X'' × X''. Denote the Fourier transforms of a in y only by

$$\tilde{a}(x,\zeta,\xi) = (2\pi)^{-n}\int e^{-iy\cdot\zeta}a(x,y,\xi)\, dy$$

and in (x,y) by

$$\hat{a}(\eta,\zeta,\xi) = (2\pi)^{-2n} \int e^{-i(x\cdot\eta\, +\, y\cdot\zeta)}a(x,y,\xi)\, dy \;=\; (2\pi)^{-n} \int e^{-ix\cdot\eta}\tilde{a}(x,\zeta,\xi)\, dx .$$

Define the following set of norms for a symbol of order p: Given $j,k,l \ge 0$, let

$$|a|_{j,k,l} = \Sigma_{|\alpha|\le l}\ \sup_{(\eta,\zeta,\xi)} |(1+|\eta|)^j(1+|\zeta|)^k(1+|\xi|)^{-p+l}(\partial_{\xi})^{\alpha}\hat{a}(\eta,\zeta,\xi)| .$$

Given a symbol $a(x,y,\xi)$, we define the following (generalized) pseudodifferential operator:

$$a(x,D)u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\, dy\, d\xi .$$

It is not difficult to show that the a "generalized" pseudodifferential operators consists merely of a classical pseudodifferential operator plus an infinitely smoothing operator (see [Ta], [Tr], or [Hö2]).

(B.1.3)Lemma. Let $a(x,y,\xi)$ be a symbol of order p and $u(x)$ a smooth compactly supported function on R^n. Then the following estimate holds:

$$\|a(x,D)u\|_{2,s} \le C|a|_{s+n+1,s+n+1+|p|,0}\,\|u\|_{2,s+p}\,,\quad s \in R^+,$$

where the constant C depends only on n.

Proof. Using the fact that the Fourier transform of the product of two functions is the convolution of the Fourier transforms, we obtain

$$\begin{aligned} a(x,D)u(x) &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,dy\,d\xi \\ &= \int e^{ix\cdot\xi}\tilde{a}(x,\xi-\zeta,\xi)\hat{u}(\zeta)\,d\zeta\,d\xi. \end{aligned}$$

Now take the Fourier transform of both sides and switch the order of integration (noting that the integral is absolutely convergent):

$$\begin{aligned} \widehat{a(x,D)u}(\eta) &= (2\pi)^{-n}\int e^{-ix\cdot(\eta-\xi)}\quad \tilde{a}(x,\xi-\zeta,\xi)\hat{u}(\zeta)\,d\zeta\,d\xi\,dx \\ &= \int \hat{a}(\eta-\xi,\,\xi-\zeta,\,\xi)\hat{u}(\zeta)\,d\zeta\,d\xi\quad. \end{aligned}$$

Therefore,

$$(1+|\eta|)^s|\widehat{a(x,D)u}(\eta)| \le |a|_{j,k,0}\int (1+|\eta|)^s\,(1+|\eta-\xi|)^{-j}\,(1+|\xi-\zeta|)^{-k}\,(1+|\xi|)^p\,|\hat{u}(\zeta)|\,d\zeta\,d\xi\;.$$

Observe that $(1+|\eta|)^s \le (1+|\eta-\xi|+|\xi-\zeta|+|\zeta|)^s \le (1+|\eta-\xi|)^s(1+|\xi-\zeta|)^s(1+|\zeta|)^s$, and by a similar argument,

$$(1+|\xi|)^{|p|} \le (1+|\xi-\zeta|)^{|p|}(1+|\zeta|)^{|p|}\quad\text{and}\quad (1+|\zeta|)^{|p|} \le (1+|\xi-\zeta|)^{|p|}(1+|\xi|)^{|p|},$$

so that whether p is positive or negative, $(1+|\xi|)^p \le (1+|\xi-\zeta|)^{|p|}(1+|\zeta|)^p$. It then follows that

$$(1+|\eta|)^{s}|\widehat{a(x,D)u}(\eta)| \le |a|_{j,k,0}\int (1+|\eta-\xi|)^{s-j}(1+|\xi-\zeta|)^{s+|p|-k}(1+|\zeta|)^{s+p}|\hat{u}(\zeta)|\, d\zeta\, d\xi .$$

Taking the L^2 norm of both sides and applying Young's inequality, we get

$$\|a(x,D)u\|_{2,s} \le C_n|a|_{j,k,0}\|u\|_{2,s+p} ,$$

if $s\text{-}j = s + |p| - k = -n\text{-}1 \Rightarrow j = s + n + 1$ and $k = s + |p| + n + 1$.

Q.E.D.

B.2 A little symplectic geometry

Recall that for any smooth domain $X \subset R^n$, the cotangent bundle of X, $T^*X \cong X \times R^n$, has a **canonical 1-form**, denoted θ, characterized as follows:

$$\langle\theta(x,\xi), V\rangle = \langle\xi, \pi_* V\rangle , \quad V \in T_{(x,\xi)}T^*X, \ (x,\xi) \in T^*X \quad .$$

If $x^1,\dots, x^n$ are local co-ordinates on X, then they induce local co-ordinates $x^1,\dots,x^n$, $\xi_1,\dots, \xi_n$ on T^*X and the canonical 1-form is $\theta = \xi_i\, dx^i$. The exterior derivative of θ, $d\theta = d\xi_i \wedge dx^i$ is a closed, nondegenerate 2-form on T^*X called the **symplectic form**. It induces an isomorphism

$$\text{(B.2.1)} \qquad T_xX \cong T^*_xX, \ x \in X, \qquad V \longmapsto i(V)d\theta,$$

where $i(V)d\theta$ denotes the 1-form obtained by contracting the tangent vector V with the 2-form $d\theta$.

Denote $T^*X\backslash\{0\} = \{ (x,\xi) \in T^*X \mid \xi \neq 0 \}$. A smooth map C: $T^*X\backslash\{0\} \to T^*X\backslash\{0\}$ is called a **canonical transformation** if $C^*\theta = \theta$. In co-ordinates, $(y,\eta) = C(x,\xi)$ is canonical if it satisfies the following system of PDE's:

$$\eta_j\frac{\partial y^j}{\partial x^i} = \xi_i \quad ; \quad \eta_j\frac{\partial y^j}{\partial \xi_i} = 0 , \quad 1 \le i \le n .$$

Observe that a canonical transformation must be an immersion since $(d\theta)^n$ is a volume form on T^*X and $C^*(d\theta)^n = (d\theta)^n$. The map C must also be homogeneous in the following sense:

lemma. A canonical transformation $C: T^*X\backslash\{0\} \to T^*X\backslash\{0\}$ satisfies: $C(x,t\,\xi) = (y, t\eta)$ for all $t > 0$, where $C(x,\xi) = (y,\eta)$. In other words, as a function of ξ, $y(x,\cdot)$ is homogeneous of degree 0 and $\eta(x,\cdot)$ is homogeneous of degree 1.

proof. This is easy and we just indicate the idea. Since the map C preserves θ and $d\theta$, it also preserves the dual vector field to θ, which is $\xi_i(\partial/\partial\xi_i)$. This is the radial vector field in the fiber of T^*X and represents an infinitesimal dilation in the ξ co-ordinate. In particular, a function $f(x,\xi)$ is homogeneous of degree p if and only if $\xi_i(\partial/\partial\xi_i)f = pf$. From these observations the lemma can be proved.

q.e.d.

In this paper we only care about canonical transformations of T^*X'', where X'' is a simply connected bounded domain of R^n. It is ncessary to "parameterize" the space of all canonical transformations near the identity. This is best done using generating (or phase) functions. Also, the phase function is a basic object in the definition of a Fourier integral operator. The prototype is $\varphi_0(x,\eta) = x^i\eta_i$, which is what appears in the exponential factor of a pseudodifferential operator. We shall consider only phase functions that are pertubations of φ_0. Accordingly, a **generating** or **phase function** is a function $\varphi(x,\eta)$ such that $\varphi - \varphi_0 \in \mathcal{S}_h^{1,\infty}$ and that the following inequality holds:

$$\text{(B.2.2)} \qquad \det \frac{\partial^2\varphi}{\partial x\partial\eta}(x,\eta) \neq 0\,, \quad (x,\eta) \in T^*X\backslash\{0\}\ .$$

Let $\mathcal{P}^\infty$ denote the space of smooth phase functions and $\mathcal{P}^{j,k}$ the completion of $\mathcal{P}^\infty$ with respect to the symbol norm $\| \|_{j,k}$. Observe that for fixed j , $\{\mathcal{P}^{j,k}\}$ is a tame scale of Banach spaces. Using a generating function φ, we try to define implicitly a map C_φ: $T^*X\backslash\{0\} \to T^*\mathbf{R}^n\backslash\{0\}$ as follows:

$$(y,\eta) = C_\varphi(x,\xi) \Leftrightarrow \xi = \frac{\partial\varphi}{\partial x} \text{ and } y = \frac{\partial\varphi}{\partial\eta} \quad .$$

Without some further assumption, the map C_φ may not be globally well-defined. When $\varphi = \varphi_0(x,\eta) = x^i\eta_i$, the corresponding map is the identity map. For phase functions sufficiently close to φ_0 in the C^2-norm, the map C_φ is still well-defined and is canonical. We leave the proof as an easy exercise. Observe that since $\varphi = \varphi_0$ outside a compact subset of X'', C_φ preserves a neighborhood of the boundary of $X''\times\mathbf{R}^n$.

B.3 Classical Fourier integral operators

Let us begin by briefly describing the "modern" Fourier integral operator. It is a linear operator F acting on functions on a manifold X such that in local co-ordinates, it can be written in the following form:

$$Fu(x) = \int e^{i\varphi(x,y,\theta)}a(x,y,\theta)u(y) \; dy \, d\theta,$$

where $\varphi(x,y,\theta)$ is a generalized phase function and $a(x,y,\theta)$ is a generalized symbol. It can be shown that if a is of order p, then F is a bounded map of H^{k+p} to H^k. The generalized phase function φ determines a Lagrangian submanifold of $T^*X\backslash\{0\} \times T^*X\backslash\{0\}$ (roughly speaking, a submanifold of $T^*X\backslash\{0\} \times T^*X\backslash\{0\}$ is **Lagrangian** if and only if it is locally the graph of a canonical transformation) .

The canonical transformation determined by φ controls how the singularities of u are transformed by F. In particular, the singular set (or, to be more precise, the wavefront set) of

Fu is contained in the image under the canonical transformation of the singular set (or wavefront set) of u. The significance of this lies in the fact that any two differential or pseudodifferential operators which have the same characteristics can be made to look alike modulo infinitely smoothing operators by conjugating with appropriately chosen pseudodifferential operators. Therefore, by conjugating a given differential operator with both pseudodifferential and Fourier integral operators, we can try to put the differential operator into a normal form which is determined solely by the qualitative behavior of the characteristics of the operator. Usually, this can only be done "microlocally", in other words, on a conic open subset of the cotangent bundle. For example, Egorov's theorem states that a differential operator of real principal type is microlocally equivalent to the differential operator $\partial/\partial x^n$. We show in this paper that any two sufficiently close differential operators of strong real principal type are globally equivalent under conjugation by Fourier integral operators.

We now restrict our attention to what we call "classical Fourier integral operators". Such operators are in fact the precursors of the modern Fourier integral operator as developed in [Hö1] (see [Hö1] p. 80, for a brief history). Given a phase function $\varphi(x,\eta)$ and a symbol $a(x,\eta)$ of order p on $\mathbf{R}^n$, define the following operator:

$$a^{\varphi}(x,D)u(x) = \int e^{i\varphi(x,\eta)}a(x,\eta)\hat{u}(\eta)\ d\eta\ .$$

Such a linear operator is called a **classical Fourier integral operator of order p.** As described above, the phase function φ determines a canonical transformation C_{φ} which describes how the operator $a^{\varphi}(x,D)$ transforms the singularities of its argument. Observe that if $\varphi(x,\eta) = \varphi_0(x,\eta)$ ($= x^i\eta_i$), then $a^{\varphi}(x,D) = a(x,D)$, a pseudodifferential operator and the canonical transformation C_{φ} is the identity map. The converse also holds: any Fourier integral operator associated with the identity map is a pseudodifferential operator. It also can be shown (see [Hö1], [Hö2], [Ta],or [Tr]) that given two Fourier integral operators,

$a^{\varphi}(x,D)$ and $b^{\Psi}(x,D)$, the composition $a^{\varphi}(x,D)b^{\Psi}(x,D)$ is also a Fourier integral operator whose associated canonical transformation is $C_{\varphi} \circ C_{\Psi}$.

It can be shown that the L^2-adjoint $a^{\varphi}(x,D)^*$ of $a^{\varphi}(x,D)$ is also a Fourier integral operator of order p whose associated canonical transformation is the inverse map to C_{φ}. In particular, the operator $a^{\varphi}(x,D)a^{\varphi}(x,D)^*$ is associated with the identity map and therefore is a pseudodifferential operator. We will compute the symbol of $a^{\varphi}(x,D)a^{\varphi}(x,D)^*$ as a pseudodifferential operator and apply lemma(B.1.1) to compute the operator norm of $a^{\varphi}(x,D)$.

Given $k \in R$, let D_k be the pseudodifferential operator such that

$$\widehat{D_k u}(\xi) = (1+|\xi|)^k \hat{u}(\xi) .$$

Observe that D_k is self-adjoint and $D_{-k}D_k = I$.

(B.3.1)Proposition. Let $\varphi(x,\eta)$ be a phase function and $a(x,\eta)$ a symbol of order $p < -n/2$ on R^n. Then $a^{\varphi}(x,D)a^{\varphi}(x,D)^*$ is a pseudodifferential operator of order $2p$. Moreover, there exists $\delta > 0$ such that if $\|\varphi - \varphi_0\|_{n+1,1} < \delta$, there is the following explicit formula:

$$a^{\varphi}(x,D)a^{\varphi}(x,D)^* = A(x,D)$$

Here,

$$A(x,y,\xi) = a(x, \eta(x,y,\xi))\overline{a}(y, \eta(x,y,\xi))\left|\frac{\partial \varphi'}{\partial \eta}(x,y,\xi)\right|^{-1},$$

$$\varphi'(x,y,\eta) = \int_0^1 \frac{\partial \varphi}{\partial x}(x + t\,(y-x),\eta)\,dt,$$

and $\eta(x,y,\xi)$ is determined uniquely by the equation $\xi = \varphi'(x,y,\eta)$.

Proof. The L^2-adjoint of $a^{\varphi}(x,D)$ is

$$\widehat{a^{\varphi}(x,D)^* v}(\eta) = \int e^{-i\varphi(x,\eta)}\, \bar{a}(x,\eta)\, v(x)\ dx \ .$$

Therefore,

$$\begin{aligned} a^{\varphi}(x,D)a^{\varphi}(x,D)^*u(x) &= \int e^{i(\varphi(x,\eta) - \varphi(y,\eta))} a(x,\eta)\bar{a}(y,\eta)u(y)\ dy\ d\eta \\ &= \int e^{i(x-y)\cdot\varphi'(x,y,\eta)}\ a(x,\eta)\bar{a}(y,\eta)u(y)\ dy\ d\eta \ . \end{aligned}$$

Now observe that $\varphi_0'(x,y,\eta) = \eta$. Therefore, given $\|\varphi-\varphi_0\|_{1,1} < \delta < 1$,

$$\left\|\frac{\partial\varphi'}{\partial\eta} - 1\right\|_0 = \left\|\int_0^1 \frac{\partial^2}{\partial x\partial\eta}(\varphi-\varphi_0)(y + s(x-y))\ ds\right\|_0 \le \|\varphi - \varphi_0\|_{1,1} < \delta \ .$$

By the implicit function theorem (the easy one!), there exists a unique smooth function $\eta(x,y,\xi)$ close to $\eta_0(x,y,\xi) = \xi$ such that $\xi = \varphi'(x,y,\eta(x,y,\xi))$. Now, since $|a(x,\eta)\bar{a}(y,\eta)| \le (1+|\eta|)^{2p}$, $2p < -n$, the integral is absolutely convergent. We can therefore change the order of integration, change the variable of integration from η to ξ, and then change the order back. This results in the desired formula.

Q.E.D.

(B.3.2)Corollary. Given a phase function $\varphi(x,\eta)$ satisfying the assumptions of Proposition(B.3.1), a symbol $a(x,\eta)$ of order p, and a smooth compactly supported function u,

$$\|a^{\varphi}(x,D)u\|_0 \le C(\|A\|_{n+1,3n/2+1,0})^{1/2}\|u\|_p \ ,$$

where the constant C depends only on n and $\delta > 0$ (as given by Proposition(B.3.1)).

Proof. First consider the case $p < -n/2$. Observe that the operator norm of $a^{\varphi}(x,D)$ is the same as the operator norm of $a^{\varphi}(x,D)^*$. On the other hand, by the proposition,

$$\begin{aligned} \|a^{\varphi}(x,D)^*u\|_0^2 = \langle u,a^{\varphi}(x,D)a^{\varphi}(x,D)^*u\rangle &= \langle u,A(x,D)u\rangle \\ &\le C\|A\|_{n+1,|p|+n+1,0}\|u\|_p^2 \ . \end{aligned}$$

Therefore, if $p = -n/2 - 1$

:

$$\|a^{\varphi}(x,D)u\|_0 \le C'(\|A\|_{n,3n/2+1,0})^{1/2}\|u\|_{-n/2-1} .$$

Now take $p \ge -n/2$. First, observe that $a^{\varphi}(x,D)D_{-p-n/2-1} = \tilde{a}^{\varphi}(x,D)$, where $\tilde{a}(x,\eta) = (1+|\eta|)^{-p-n/2-1}a(x,\eta)$ is a symbol of order -n/2-1. We then have:

$$\|a^{\varphi}(x,D)u\|_0 = \|a^{\varphi}(x,D)D_{-p-n/2-1}D_{p+n/2+1}u\|_0$$

$$= \|\tilde{a}^{\varphi}(x,D)(D_{p+n/2+1})u\|_0 \le C(\|\tilde{A}\|_{n,3n/2+1,0})^{1/2}\|D_{p+n/2+1}u\|_{-n/2-1}$$

$$= C(\|A\|_{n,3n/2+1,0})^{1/2}\|u\|_p .$$

Q.E.D.

B.4 Moser-type estimates

We now want to prove Moser-type estimates for pseudodifferential and Fourier integral operators like those proved for linear differential operators in §A.5. Since a pseudodifferential operator is simply a Fourier integral operator whose phase function is φ_0, it suffices to consider only Fourier integral operators. As in the proof of lemma(A.5.2), we proceed by induction. The key point is what happens when we differentiate $a^{\varphi}(x,D)u(x)$ and, as before, the product rule for differentiation ("integration by parts"!) plays a key role.

(B.4.1)Theorem. Let $\delta > 0$ be as given by Proposition(B.3.1) and fix $B > 0$. There exists $\alpha \ge n+1$ such that given any phase function φ satisfying $\|\varphi-\varphi_0\|_{n+1,1} < \delta$ and $\|\varphi\|_{\alpha+1,0} < B$ and symbol $a(x,\eta)$ of order p, the following estimate holds:

$$\|a^{\varphi}(x,D)u\|_s \le C_s[\,(\|a\|_{\alpha+s,0} + \|a\|_{\alpha,0}\|\varphi\|_{\alpha+s+1,0})\|u\|_p + \|a\|_{\alpha,0}\|u\|_{s+p}\,] \ , \ s \ge 0,$$

where the constant C_s depends only on n, δ, B, and s.

In particular the map $(\varphi, a, u) \longmapsto a^{\varphi}(x,D)u$ is a smooth tame map for all (φ,a,u) such that φ lies in the open neighborhood of φ_0 given by the assumptions.

Proof. In the following estimates, the constant C changes from line to line. In particular, it will be "absorbing" $\|\varphi\|_{\alpha+1,0}$ whenever this norm appears.

First, by Corollary(B.3.2), we can choose $\alpha' > 0$ so that

$$\|a^{\varphi}(x,D)u\|_0 \le C\|a\|_{\alpha,0}\|u\|_p.$$

Set $\alpha = \alpha' + 1$. Observe that if we differentiate $a^{\varphi}(x,D)u$, we get

$$\partial_j a^{\varphi}(x,D)u(x) = (\partial_j a)^{\varphi}(x,D)u + (ia\partial_j\varphi)^{\varphi}(x,D)u,$$

where $(\partial_j a)^{\varphi}(x,D)$ is a Fourier integral operator of order p and $(ia\partial_j\varphi)^{\varphi}(x,D)$ is of order p + 1. Consider

$$\begin{aligned}\|\partial_j a^{\varphi}(x,D)u\|_0 &\le \|(\partial_j a)^{\varphi}(x,D)u\|_0 + \|(ia\partial_j\varphi)^{\varphi}(x,D)u\|_0\\ &\le C[\ \|a\|_{\alpha,0}\|u\|_p + \|a\partial_j\varphi\|_{\alpha',0}\|u\|_{p+1}\\ &\le C\|a\|_{\alpha,0}\|u\|_{p+1}.\end{aligned}$$

Now we proceed by induction. Assume that the estimate holds for $s \le k$. For convenience, we denote the symbol norm $\|\ \|_{s,0}$ by $\|\ \|_s$. Consider

$$\begin{aligned}\|\partial_j a^{\varphi}(x,D)u\|_k &\le \|(\partial_j a)^{\varphi}(x,D)u\|_k + \|(ia\partial_j\varphi)^{\varphi}(x,D)u\|_k\\ &\le C[\ (\|\partial_j a\|_{\alpha+k} + \|\partial_j a\|_{\alpha}\|\varphi\|_{\alpha+k+1})\|u\|_p + \|\partial_j a\|_{\alpha}\|u\|_{k+p}\\ &\qquad + (\|a\partial_j\varphi\|_{\alpha+k} + \|a\partial_j\varphi\|_{\alpha}\|\varphi\|_{\alpha+k+1})\|u\|_{p+1} + \|a\partial_j\varphi\|_{\alpha}\|u\|_{k+p+1}\ ]\\ &\le C[\ \|a\|_{\alpha+k+1}\|u\|_p + \|a\|_{\alpha+1}\|\varphi\|_{\alpha+k+1}\|u\|_p + \|a\|_{\alpha+1}\|u\|_{k+p}\\ &\qquad + (\ \|a\|_{\alpha}\|\varphi\|_{\alpha+k+1} + \|a\|_{\alpha+k}\|\varphi\|_{\alpha+1} + \|a\|_{\alpha}\|\varphi\|_{\alpha+1}\|\varphi\|_{\alpha+k+1})\|u\|_{p+1}\\ &\qquad \|a\|_{\alpha}\|\varphi\|_{\alpha+1}\|u\|_{k+p+1}.\end{aligned}$$

Now applying Corollary(A.5.2) to all but the first and last terms, we obtain

$$\|\partial_j a^{\varphi}(x,D)u\|_k \le C[\,(\|a\|_{\alpha+k+1} + \|a\|_{\alpha}\|\varphi\|_{\alpha+k+2})\|u\|_p + \|a\|_{\alpha}\|u\|_{k+p}].$$

Q.E.D.

Setting $\varphi = \varphi_0$, we also obtain Moser-type estimates for pseudodifferential operators:

(B.4.2)**Corollary.** The map $(a, u) \longmapsto a(x,D)u$ is a smooth tame map.

B.5 Composition of a pseudodifferential operator with a classical Fourier integral operator and vice versa

(B.5.1)**Theorem.** Let $\varphi(x,\eta)$ be a phase function, $a(x,\xi)$ a symbol of order p, and $b(x,\eta)$ a symbol of order q. Then

$$a(x,D)b^{\varphi}(x,D) = c^{\varphi}_1(x,D)$$

and

$$b^{\varphi}(x,D)a(x,D) = c^{\varphi}_2(x,D),$$

where

$$c^{\varphi}_1(x,\eta) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi}\, a(x,\varphi'(x,y,\eta)+\xi')b(y,\eta)\; dy\; d\xi \quad ,$$

and

$$c^{\varphi}_2(x,\eta) = (2\pi)^{-n} \int e^{i(\eta-\xi)\cdot y}\, b(x,\xi)a(\varphi'(x,\xi,\eta)+y,\eta)\; dy\; d\xi \quad .$$

Here,

$$\varphi'(x,y,\eta) = \int_0^1 \frac{\partial\varphi}{\partial x}(x+t(y-x),\eta)\; dt \quad ,$$

and

$$\varphi''(x,\xi,\eta) = \int_0^1 \frac{\partial\varphi}{\partial\eta}(x,\eta + s(\xi-\eta))\ dt \ .$$

(Observe that both φ' and φ'' are vector-valued.)

Remark. The integrals above and below are convergent only as iterated integrals; they are generally not absolutely convergent.

Proof.

$$a(x,D)b^{\varphi}(x,D)u(x) = \int e^{ix\cdot\xi} a(x,\xi)\Big[(2\pi)^{-n}\int e^{-i(y\cdot\xi-\varphi(y,\eta))} b(y,\eta)\hat{u}(\eta)\ d\eta\ dy\Big]\ d\xi$$

$$= \int e^{i[(x-y)\cdot\xi + \varphi(y,\eta)]} a(x,\xi)b(y,\eta)\hat{u}(\eta)\ d\eta\ dy\ d\xi \ .$$

Expand $\varphi(y,\eta) = \varphi(x,\eta) - (x-y)\cdot\varphi'(x,y,\eta)$. We now want to switch the order of integration from $d\eta\ dy\ d\xi$ to $dy\ d\xi\ d\eta$. Since the integral is not absolutely convergent, we must first integrate by parts, using the following identity:

$$(1 + |\xi|^2)^{-N}(1 - \Delta_y)^N e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi} \ ,$$

where N is a nonnegative integer and $\Delta_y = \Sigma_{1\le i\le n} (\partial/\partial y^i)^2$. By choosing N large enough, integrating by parts, we obtain a new integral which is absolutely convergent. We can then switch the order of integration and reverse the integration by parts, obtaining:

$$a(x,D)b^{\varphi}(x,D)u(x) = \int e^{i[\varphi(x,\eta) + (x-y)\cdot(\xi - \varphi'(x,y,\eta))]}\ a(x,\xi)b(y,\eta)\hat{u}(\eta)\ dy\ d\xi\ d\eta \ .$$

Setting $\xi' = \xi - \varphi'(x,y,\eta)$, we obtain the desired formula.

Now consider

$$b^{\varphi}(x,D)a(x,D)u(x) = \int e^{i\varphi(x,\xi)}\ b(x,\xi)\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)\ d\eta\ d\xi \ .$$

Expand $\varphi(x,\xi) = \varphi(x,\eta) + (\xi-\eta)\cdot\varphi''(x,\xi,\eta)$. Using the same trick as above to switch the order of integration, we then obtain

$$b^{\varphi}(x,D)a(x,D)u(x) = \int e^{i\varphi(x,\eta)} \int e^{i(\xi-\eta)\cdot\varphi''(x,\xi,\eta)}\, b(x,\xi)\hat{a}(\xi-\eta,\eta)\, d\xi \quad \hat{u}(\eta)\, d\eta$$

$$= \int e^{i\varphi(x,\eta)}\, c_2(x,\eta)\hat{u}(\eta)\, d\eta\ .$$

Q.E.D.

Using the Taylor series formula, the symbols $c_1(x,\eta)$ and $c_2(x,\eta)$ can be expanded into the usual description of the symbol calculus using an asymptotic sum. The asymptotic sum describes the composition modulo an infinitely smoothing operator. We shall need an exact expression which contains only a finite number of terms in the asymptotic expansion. This is easily obtained by using the finite Taylor expansion, writing the error term down explicitly as an integral, leading to the following: Given a multi-index α, let $D_\xi^{\alpha} = (i\partial_\xi)^{\alpha}$.

(B.5.2)**Theorem.** Let $c_1(x,\eta)$ and $c_2(x,\eta)$ be the symbols defined by Theorem(B.5.1). The following hold:

$$c_1(x,\eta) = \sum_{j=0}^{N} \frac{1}{j!} \sum_{|\alpha|=j} \partial_y^{\alpha} \Big|_{y=x} [\, D_\xi^{\alpha}\, a(x,\varphi'(x,y,\eta))b(y,\eta)\,] + r_N(x,\eta)\ ,$$

and

$$c_2(x,\eta) = \sum_{j=0}^{N} \frac{1}{j!} \sum_{|\alpha|=j} D_\xi^{\alpha} \Big|_{\xi=\eta} [\, b(x,\xi)\partial_x^{\alpha}\, a(\varphi''(x,\xi,\eta),\eta)] + s_N(x,\eta)\ ,$$

where

$$r_N(x,\eta) = \frac{(2\pi)^{-n}}{N!} \sum_{|\alpha|=N+1} \int \int_0^1 e^{i(x-y)\cdot\xi}\, (1-s)^N\, \xi'_{\alpha}\partial_\xi^{\alpha} a(x,\varphi'(x,y,\eta)+s\xi')b(y,\eta)\ ds\, dy\, d\xi'\ ,$$

and

$$s_N(x,\eta) = \frac{(2\pi)^{-n}}{N!} \sum_{|\alpha|=N+1} \int \int_0^1 e^{i(\eta-\xi)\cdot y} (1-s)^N y^\alpha b(x,\xi) \partial_x^\alpha a(\varphi''(x,\xi,\eta)+sy,\eta) \; ds\, dy\, d\xi \, ,$$

are symbols of order $p - N - 1$ and $q - N - 1$, respectively.

Proof. Let $f(s) = a(x,\varphi'(x,y,\eta) + s\xi')$. The desired formula for $c_1(x,\eta)$ is obtained by substituting in the Nth order Taylor expansion for f centered at $s = 0$ and evaluated at $s = 1$.

The formula for c_2 is obtained in a similar fashion, using $f(s) = b(\varphi''(x,\xi,\eta) + sy, \eta)$.

Q.E.D.

Setting $\varphi = \varphi_0$ and letting $N \to \infty$, we obtain the usual symbol calculus for (pseudo)differential operators.

B.6 Inverting elliptic pseudodifferential and Fourier integral operators

A symbol $a(x,\xi)$ of order p is **elliptic** if there exists a homogeneous symbol $a_0(x,\xi)$ of order p such that $a_0(x,\xi) \neq 0$, $(x,\xi) \in X' \times \mathbf{R}^n \backslash \{0\}$, and $a - a_0$ is a symbol of order $p - 1$. A pseudodifferential operator $a(x,D)$ is elliptic if the corresponding symbol $a(x,\xi)$ is. Similarly, a Fourier integral operator $a^\varphi(x,D)$ is elliptic if $a(x,\xi)$ is. Observe that if $a(x,\xi)$ is a polynomial in ξ, then we recover the standard definition of an elliptic differential operator.

It is well-known that an elliptic pseudodifferential operator $a(x,D)$ has a parametrix $b(x,D)$, i.e. an elliptic pseudodifferential operator such that $a(x,D)b(x,D) - I$ and $b(x,D)a(x,D) - I$ are infinitely smoothing operators, where I is the identity operator. An asymptotic expansion of the symbol $b(x,\xi)$ is easily constructed using the symbol calculus for the composition of pseudodifferential operators. This may be found in any standard reference on pseudodifferential operators (cf. [Ni2], [Ta], [Tr], [Hö1], [Hö2]). We do not want to recall the details again here, but simply want to observe that if we use only a finite portion of the asymptotic expansion of $b(x,\xi)$ and the explicit description of $a(x,D)b(x,D)$ and $b(x,D)a(x,D)$ given in §B.5, we obtain the following:

(B.6.1)**Proposition.** Let $a_0(x,\xi)$ be an elliptic symbol of order p. Then there exists $k > 0$, a neighborhood $D \subset \mathcal{S}^{p,k,k}$ of a_0 such that given $N > 0$, there exists a smooth tame map

$$D \longrightarrow \mathcal{S}^{p,0,0}$$
$$a(x,\xi) \longmapsto b(x,\xi),$$

where $a(x,D)b(x,D) - I$ and $b(x,D)a(x,D) - I$ are operators of order $p - N$ when acting on functions compactly supported on X'.

Using this proposition and proposition(B.3.1), it is a simple matter to construct a right parametrix for an elliptic Fourier integral operator. Given a Fourier integral operator $a^{\varphi}(x,D)$, proposition(B.3.1) states that $a^{\varphi}(x,D)a^{\varphi}(x,D)^* = A(x,D)$, where the symbol $A(x,y,\xi)$ is a smooth tame function of $a(x,\eta)$ and $\varphi(x,\eta)$. The desired right parametrix is simply

$$Q = a^{\varphi}(x,D)^* B(x,D),$$

where $B(x,D)$ is a parametrix of $A(x,D)$ as given by proposition(B.6.1). If we fix an elliptic symbol $a_0(x,\eta)$, the symbol $B(x,\xi)$ is a smooth tame function of $(\varphi(x,\eta), a(x,\eta))$ lying in a neighborhood of $(\varphi_0(x,\eta), a_0(x,\eta))$.

Finally, we want to construct a right inverse from the parametrix. It suffices to prove the following:

(B.6.2)**Proposition.** There exists $k > 0$, a neighborhood D of $0 \in \mathcal{S}^{0,k,0}$ and a smooth tame map

$$D \times H^0(\overline{X}') \longrightarrow H^0(\overline{X}')$$

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