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LIVE TRANSCRIPTION
Smooth Immersed Curves

- Recall that we have defined a smooth parameterized curve in an affine space $\mathbb{A}$ to be a smooth map
  
  $$c : I \to \mathbb{A}$$

  where $\dot{c}(t) \neq 0$ for all $t \in I$

- Note that we allow the curve to intersect itself
- Such curves are also called immersed curves
- If a curve does not intersect itself, it is called an embedded curve
Winding versus Rotation Number of a Closed Planar Curve

- **Winding number** $W(p, C)$ of a closed planar curve $C \subset \mathbb{A}^2$ around a point $p \notin C$ is the number of times the curve goes counterclockwise around $p$

  $W(p_1, C_1) = W(p_5, C_2) = 0$
  $W(p_2, C_1) = W(p_4, C_2) = 1$
  $W(p_3, C_2) = 2$

- **Rotation number** $R(C)$ is the number of times the unit tangent vector rotates counterclockwise around the circle

  $R(C_1) = 1$
  $R(C_2) = 2$
“Obvious” Facts about the Winding and Rotation Numbers

- The winding number \( W(C, p) \)
  - Depends on where \( p \) lies relative to the curve
  - Equals zero if \( p \) lies outside the curve completely
  - If \( p_1 \) and \( p_2 \) are points that can be connected by a curve that does not cross \( C \), then
    \[
    W(C, p_1) = W(C, p_2)
    \]

- If a curve \( C_1 \) can be continuously deformed through a family of closed curves into another curve \( C_2 \) without any of the curves crossing \( p \), then
  \[
  W(C_1, p) = W(C_2, p)
  \]

- The rotation number
  - Remains unchanged under any smooth deformation of the curve
A smooth curve $c : [0, T] \rightarrow \mathbb{R}^2$ can be written using polar coordinates relative to a point $p$ not on the curve as

$$c(t) = p + e_1 x(t) + e_2 y(t) = p + r(t)(e_1 \cos(\theta(t)) + e_2 \sin(\theta(t))),$$

where $r(t)$ is always nonzero.

Differentiating this, we get

$$e_1 \dot{x} + e_2 \dot{y} = \dot{r}(e_1 \cos \theta + e_2 \sin \theta) + \dot{\theta}(-e_1 r \sin \theta + e_2 r \cos \theta)$$

$$= \frac{\dot{r}}{r}(e_1 x + e_2 y) + \dot{\theta}(-e_1 y + e_2 x).$$

Therefore,

$$\dot{\theta} = \frac{-y \dot{x} + x \dot{y}}{x^2 + y^2}$$

and

$$\theta(T) - \theta(0) = \int_{t=0}^{t=T} \frac{-y \dot{x} + x \dot{y}}{x^2 + y^2} \, dt.$$
Winding Number of a Closed Curve

- If $c : [0, T] \to \mathbb{R}^2$ is a closed curve and $p$ does not lie on the curve, then

$$c(0) = c(T) \implies x(0) = x(T) \text{ and } y(0) = y(T)$$

$$\implies r(0) = r(T) \text{ and } \theta(T) - \theta(0) = 2\pi k, \text{ for some integer } k$$

- Therefore,

$$\frac{1}{2\pi} \int_{t=0}^{t=T} \frac{-y\dot{x} + x\dot{y}}{x^2 + y^2} \, dt = \frac{1}{2\pi} \int_{t=0}^{t=T} \dot{\theta}(t) \, dt = \theta(T) - \theta(0) = k$$

- Equivalently, the line integral

$$\frac{1}{2\pi} \int_{C} \frac{-y \, dx + x \, dy}{x^2 + y^2}$$

is always an integer and equal to the winding number.
Winding Number is a Topological Invariant

- Suppose $c_\delta : [0, 1] \rightarrow \mathbb{E}^2$ is a continuous family of closed curves, parameterized by $0 \leq \delta \leq 1$.

- In other words, for each $0 \leq \delta \leq 1$, the curve $c_\delta$ satisfies

  \[ c_\delta(0) = c_\delta(1) \]

- If we define the polar angle $\theta$ such that for each $0 \leq \delta \leq 1$,

  \[ \theta_\delta(0) = 0 \]

  then

  \[ \theta_\delta(1) = 2\pi k_\delta \]

- On the other hand,

  \[ \theta_\delta(1) - \theta_\delta(0) = \int_{t=0}^{t=1} \frac{-y_\delta \dot{x}_\delta + x_\delta \dot{y}_\delta}{x_\delta^2 + y_\delta^2} \, dt \]

  is a continuous function of $\delta$.

- Therefore, the winding number $W(C_\delta, p) = k_\delta$ is a constant independent of $\delta$. 
Frenet-Serret Frame and Equations for Parameterized Curve in $\mathbb{E}^2$

The Frenet-Serret frame for a parameterized curve $c : I \rightarrow \mathbb{E}^2$ is an oriented orthonormal frame $F = (f_1, f_2)$ along $c$ such that

$$c' = \sigma f_1$$

The Frenet-Serret equations are

$$\frac{1}{\sigma} \frac{d}{dt} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix},$$

where $\kappa$ is the curvature function.
Rotation Angle of a Parameterized Curve

- Fix an orthonormal basis \((e_1, e_2)\) of \(\mathbb{V}^2\)
- Consider a curve \(c: I \to \mathbb{E}^2\) with Frenet-Serret frame \((f_1, f_2)\)
- The counterclockwise angle \(\phi\) from \(e_1\) to \(f_1\) satisfies

\[
\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}
\]
Curvature is Normalized Rate of Change of Angle

- On one hand, the Frenet-Serret equations say

\[
\frac{d}{dt} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix} \sigma
\]

- On the other hand,

\[
\frac{d}{dt} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \begin{bmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \end{bmatrix} \phi
\]

\[
= \begin{bmatrix} -e_1 \sin \phi + e_2 \cos \phi & -e_1 \cos \phi - e_2 \sin \phi \end{bmatrix} \phi
\]

\[
= \begin{bmatrix} f_2 & -f_1 \end{bmatrix} \phi
\]

\[
= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} 0 & -\dot{\phi} \\ \phi & 0 \end{bmatrix}
\]

- Therefore,

\[
\kappa = \frac{\dot{\phi}}{\sigma} \text{ or } \dot{\phi} = \sigma \kappa
\]
Rotation Number of a Smooth Closed Curve

- If a curve \( c : [0, T] \to \mathbb{A}^2 \) is closed, then

\[ c(0) = c(T) \]

- If a closed curve is smooth and oriented in the direction \( \dot{c} \), then since \( \dot{c}(0) \) and \( \dot{c}(T) \) have the same orientation, they have to point in the same direction.

- Therefore,

\[ \phi(T) - \phi(0) = 2\pi k, \]

where \( k \) is the rotation number of \( C \).

- Since

\[ \dot{\phi} = \kappa \sigma, \]

the rotation number of \( C \) is equal to

\[ R(C) = \frac{1}{2\pi} \int_{t=0}^{t=T} \kappa(t)\sigma(t) \, dt \]
Rotation Number is a Topological Invariant

- If $c_\delta$ is a continuous family of curves parameterized by $\delta \in [0, 1]$ such that the curvature function $\kappa_\delta$ and speed function $\sigma_\delta$ are continuous functions of $\delta$, then

$$R(C_\delta) = \frac{1}{2\pi} \int_{t=0}^{t=T} \kappa_\delta(t)\sigma_\delta(t) \, dt$$

is a continuous function of $\delta$

- Since $R(C_\delta)$ is an integer, it must therefore be constant