MATH-GA2120 Linear Algebra II

Polar Decomposition
Moore-Penrose Pseudoinverse

Deane Yang

Courant Institute of Mathematical Sciences
New York University

April 25, 2024

Polar Decomposition of Linear Map

- Let X and Y be inner product spaces such that $\dim(X) = \dim(Y)$
- Consider a linear map

$$L : X \to Y$$

- Then there exists a unitary map $U : X \to Y$ such that

$$L = U|L|$$

- Proof: By the singular value decomposition of L,

$$L = W\Sigma V^* = (WV^*)V\Sigma V^* = U|L|$$

System of Linear Equations

- Consider a system of n equations with m unknowns,

$$a^1_1 x^1 + \cdots + a^1_m x^m = y^1$$
$$\vdots$$
$$a^n_1 x^1 + \cdots + a^n_m x^m = y^n$$

- Usually, there is no solution
- And, even if there is a solution, it is usually not unique
- Basic examples
  - 1 equation in 1 unknown

$$3x = 1$$

  - 1 equation in 2 unknowns

$$x + y = 1$$

  - 2 equations in 2 unknowns

$$x + y = 1$$
$$x + y = 2$$

Matrix Equation

- Given $A \in \mathcal{M}_{n\times m}(\mathbb{C})$ and $y \in \mathbb{C}^n$, we want to solve for $x \in \mathbb{C}^m$ such that
$$Ax = y$$
- The matrix A defines a map $A : \mathbb{C}^m \to \mathbb{C}^n$
- There is a solution if and only if $y \in \operatorname{image} A$
- If a solution exists, then it is unique if and only if $\ker A = \{0\}$
- It is possible that $y \notin \operatorname{image} A$, because A and y are from inexact measurements
- Instead, we look for best possible approximation

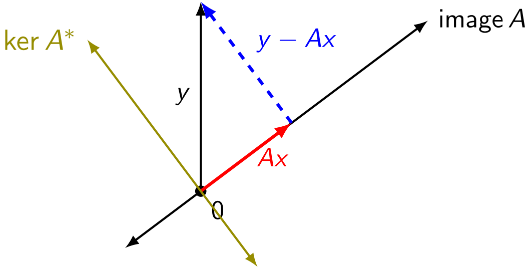
Quasi-Solution with Least Error

- Given $x \in \mathbb{C}^m$, define the error to be

$$\epsilon = L(x) - y \in \mathbb{C}^n$$

- Goal: Solve for x that minimizes the magnitude of the error, $\|\epsilon\|$
- An $x \in X$ that minimizes $\|\epsilon\|$ is called a **quasi-solution**
- If $y \in \text{image}\, L$, then a quasi-solution is a solution
- A quasi-solution need not be unique

Geometric Perspective



- If Ax is closest to y, then
  - $y - Ax$ is orthogonal to image A
- Recall that $(\text{image } A)^\perp = \ker A^*$
- Therefore, A is closest to y if

$$A^*(y - Ax)) = 0$$

or, equivalently,

$$A^*Ax = A^*y$$

Example

- Consider the system of equations

$$\begin{aligned} x + y + z &= 3 \\ x + y &= 3 \\ z &= 3 \end{aligned}$$

- Equivalently,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

- There is no solution

Quasi-Solution

- Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (x, y, z) is a quasi-solution if

$$A^* A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^* \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

Quasi-Solution Via Row Reduction

- (x, y, z) is a quasi-solution if

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\implies x + y = 2$$

$$z = 2$$

Quasi-Solution Error

- $$\begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix} \text{ is a quasi-solution to } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$
- The error of the quasi-solution

$$\epsilon = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Error Comparison

- The error for any other (x, y, z) is

$$\epsilon = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} x + y + z - 3 \\ x + y - 3 \\ z - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} x + y + z - 4 \\ x + y - 2 \\ z - 2 \end{bmatrix}$$

- The error magnitude squared is

$$\epsilon^2 = \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} x + y + z - 4 \\ x + y - 2 \\ z - 2 \end{bmatrix} \right\|^2 \geq \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\|^2$$

Quasi-Solutions of $L(x) = y$

- $L(x)$ is closest to y if

$$L^* L(x) = L^*(y)$$

- For any $y \in Y$, there is always a quasi-solution x, because

$$\text{image}(L^* L) = \text{image}\, L^*$$

  - Recall that $\ker(L^* L) = \ker L$
  - Therefore, since $L^* L$ is self-adjoint,

$$\text{image}(L^* L) = (\ker L^* L)^\perp = (\ker L)^\perp = \text{image}\, L^*$$

- If $v \in \ker L^* L = \ker L$, then $x + v$ is also a solution
- The quasi-solution is unique only if $\ker L = \{0\}$
  - Because the domain and range of $L^* L$ have the same dimension
  - If $\dim X > \dim Y$, this is not possible, because

$$\dim \ker L = \dim X - \dim(\text{image}\, L) \geq \dim X - \dim Y > 0$$

Error Comparison

- A quasi-solution of the equation $L(x) = y$ satisfies

$$L^*L(x) = L^*(y)$$

  and therefore $L^*(L(x) - y) = 0$
- The error of the quasi-solution x is

$$\epsilon = L(x) - y$$

- The error of any $x' \in X$ is

$$\epsilon' = L(x') - y = L(x' - x) + L(x) - y = L(x' - x) + \epsilon$$

- On the other hand,

$$\begin{aligned} \langle L(x' - x), \epsilon \rangle &= \langle x' - x, L^*(\epsilon) \rangle \\ &= \langle x' - x, L^*L(x) - L^*(y) \rangle \\ &= 0 \end{aligned}$$

- Therefore, $\|\epsilon'\|^2 = \|\epsilon\|^2 + \|L(x' - x)\|^2$

Quasi-Solution when $L^*L : X \to X$ is Invertible

- If x is a quasi-solution, then

$$L^*L(x) = L^*(y)$$

- If the map $L^*L : X \to X$ is invertible, then the unique quasi-solution is

$$x = (L^*L)^{-1}L^*(y)$$

Solution with Minimal Magnitude

- Suppose $x \in X$ is a solution (not just a quasi-solution) of

$$Ax = y$$

- If $v \in \ker A$, then $x + v$ is also a solution,

$$A(x + v) = y$$

- There is a unique solution x with minimal magnitude

Minimal Magnitude Solution Via Orthogonal Projection

- For any $x' \in X$, there is a unique way to decompose x' into a sum

$$x' = x + (x' - x),$$

where $x \in (\ker A)^{\perp}$ and $x - x' \in \ker A$

- If x' is a solution to

$$Ax' = y,$$

then

$$Ax = A(x - x') + Ax' = y$$

- If $x_1, x_2 \in (\ker A)^{\perp}$ are both solutions, then

$$x_1 - x_2 \in (\ker A)^{\perp} \text{ and } x_1 - x_2 \in \ker A,$$

because

$$A(x_1 - x_2) = Ax_1 - Ax_2 = y - y = 0$$

Therefore, $x_1 - x_2 = 0$

Quasi-Solution with Minimal Magnitude

- A quasi-solution to
$$Ax = y$$
is a solution of
$$A^*Ax = A^*y$$
- There is a unique quasi-solution $x \in (\ker A^*A)^\perp = (\ker A)^\perp$

Example

- The quasi-solutions of the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

  are

$$\begin{bmatrix} x \\ 2 - x \\ 2 \end{bmatrix}, \;\; x \in \mathbb{C}$$

- The magnitude squared of each quasi-solution is

$$\left\| \begin{bmatrix} x \\ 2 - x \\ 2 \end{bmatrix} \right\|^2 = x^2 + (2 - x)^2 + 4 = 2((x - 1)^2 + 3)$$

- The magnitude is minimized when $x = 1$ and therefore the Moore-Penrose quasi-solution is $(1, 1, 2)$

Moore-Penrose Quasi-Inverse Operator

- Let X and Y be inner product spaces and $L : X \to Y$ be a linear map
- There is a map $L^+ : Y \to X$ such that for any $y \in Y$, $x = L^+(y)$ is the unique quasi-solution with minimal magnitude of the equation

$$L(x) = y$$

- The map L^+ is called the **Moore-Penrose quasi-inverse** of L

Moore-Penrose Quasi-Inverse Operator

- The map

$$L|_{(\ker L)^\perp} : (\ker L)^\perp \to \operatorname{image} L$$

is an isomorphism.

- Let

$$\pi : Y \to \operatorname{image} L$$

be orthogonal projection

- The Moore-Penrose quasi-inverse operator is the map

$$L^+ : Y \to X,$$

given by

$$L^+(y) = \left(L|_{(\ker L)^\perp} \right)^{-1} (\pi(y)) \in (\ker L)^\perp \subset X$$

Quasi-Inverse of Diagonal Matrix

- Let $\Sigma : \mathbb{R}^m \to \mathbb{R}^m$ be the diagonal matrix such that for each $1 \leq k \leq m$,

$$\Sigma(\epsilon_k) = \begin{cases} s_k \epsilon_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases}$$

- Therefore,

$$\Sigma(\epsilon_1 v^1 + \cdots + \epsilon_m v^m) = \epsilon_1 s_1 v^1 + \cdots + \epsilon_r s_r v^r$$

- The quasi-inverse of Σ satisfies the following:

$$\Sigma^+(\epsilon_1 v^1 + \cdots + \epsilon_m v^m) = \epsilon_1 s_1^{-1} v^1 + \cdots + \epsilon_r s_r v^r$$

- In particular,

$$\begin{aligned} \Sigma^+(\Sigma(\epsilon_1 v^1 + \cdots + \epsilon_m v^m)) &= \Sigma^+(\epsilon_1 s_1 v^1 + \cdots + \epsilon_r s_r v^r) \\ &= \epsilon_1 v^1 + \cdots + \epsilon_r v^r \\ &= \pi_r(v), \end{aligned}$$

  where $\pi_r : \mathbb{R}^m \to \mathbb{R}^m$ is orthogonal projection onto the subspace spanned by $(\epsilon_1, \ldots, \epsilon_r)$

Quasi-Inverse Via Singular Value Decomposition

- Let the singular value decomposition of $L : X \to Y$ be

$$L = W\Sigma V^*,$$

- For each $1 \le k \le m$, let $e_k = V(\epsilon_k)$
- For each $1 \le j \le n$, let $f_j = W(\epsilon_j)$
- Then for any $x = e_1x^1 + \cdots + e_mx^m \in X$,

$$L(x) = L(e_1x^1 + \cdots + e_mx^m) = f_1s_1x^1 + \cdots + f_rs_rx^r$$

- Therefore, for any $y = f_1y^1 + \cdots + f_ny^n \in Y$,

$$L^+(y) = L^+(f_1y^1 + \cdots + f_ny^n) = e_1s_1^{-1}y^1 + \cdots + e_rs_r^{-1}y^r$$

- In other words,

$$L^+ = W\Sigma^+V^*$$