MATH-GA1002 Multivariable Analysis

Alternating Tensors
Differential Forms
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Differential Forms in Coordinates

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Alternating Tensors

- Let V be an m-dimensional vector space
- A 1-**tensor** is a linear function $\theta : V \to \mathbb{R}$, i.e., an element of V^*
- A 2-**tensor** is a bilinear function $\Theta : V \times V \to \mathbb{R}$
- A 2-tensor Θ is **alternating** if for any $v_1, v_2 \in V$,

$$\Theta(v_2, v_1) = -\Theta(v_1, v_2)$$

- A k-**tensor** is a multilinear function of k vectors,

$$\Theta : V \times \cdots \times V \to \mathbb{R}$$

- A k-tensor Θ is **alternating** if for any permutation $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$,

$$\Theta(v_{\sigma(1)}, \ldots, v_{\sigma(k)} = \epsilon(\sigma)\Theta(v_1, \ldots, v_m)$$

- Let $\Lambda^k V^*$ denote the space of all k-tensors, which is a vector space

Differential Forms

- Let $\Lambda_x^k O = \Lambda^k T_x^* O$ and

$$\Lambda^k O = \coprod_{x \in O} \Lambda_x^k O$$

- A **differential** k**-form** is a map

$$\Theta : O \to \Lambda^k O$$

such that for each $x \in O$, $\Theta(x) \in \Lambda_x^k O$

Tensor and Wedge Product

- The **tensor product** of $\theta^1, \theta^2 \in V^*$ is the 2-tensor $\theta^1 \otimes \theta^2$, where
$$(\theta^1 \otimes \theta^2)(v_1, v_2) = \langle \theta^1, v_1 \rangle \langle \theta^2, v_2 \rangle$$
- The **wedge product** of $\theta^1, \theta^2 \in V^*$ is the alternating 2-tensor $\theta^1 \wedge \theta^2 \in \Lambda^2 V^*$ given by
$$\theta^1 \wedge \theta^2 = \theta^1 \otimes \theta^2 - \theta^2 \wedge \theta^1$$
or, equivalently,
$$(\theta^1 \wedge \theta^2)(v_1, v_2) = \langle \theta^1, v_1 \rangle \langle \theta^2, v_2 \rangle - \langle \theta^2, v_1 \rangle \langle \theta^1, v_2 \rangle$$
- The **wedge product** of $\theta^1, \ldots, \theta^k \in V^*$ is the alternating k-tensor $\theta^1 \wedge \cdots \wedge \theta^k$, where
$$(\theta^1 \wedge \cdots \wedge \theta^k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \langle \theta^{\sigma(1)}, v_1 \rangle \cdots \langle \theta^{\sigma(k)} v_k \rangle$$

Basic Properties of Wedge Product

- If $\tau \in S_k$, then
$$\theta^{\tau(1)} \wedge \cdots \wedge \theta^{\tau(k)} = \epsilon(\sigma)\theta^1 \wedge \cdots \wedge \theta^k$$
- If for some $1 \leq i < j \leq k$, $\theta^i = \theta^j$, then
$$\theta^1 \wedge \cdots \wedge \theta^k = 0$$
- If $\theta^1, \ldots, \theta^k$ are linearly dependent, then
$$\theta^1 \wedge \cdots \wedge \theta^k = 0$$
- If $\theta^j = A^j_i \phi^i$, then
$$\theta^1 \wedge \cdots \wedge \theta^k = (\det(A))\phi^1 \wedge \cdots \wedge \phi^k$$

Pullback of Tensors

- If $\Theta \in \Lambda^2 W^*$, then $L^*\Theta \in \Lambda^2 V^*$ is defined by

$$L^*\Theta(v_1, v_2) = \Theta(L(v_1), L(v_2))$$

- If $\Theta \in \Lambda^k W^*$, then $L^*\Theta \in \Lambda^k V^*$ is defined by

$$L^*\Theta(v_1, v_2, \ldots, v_k) = \Theta(L(v_1), L(v_2), \ldots, L(v_k))$$

Pullback of Wedge Product

- If $\theta^1, \ldots, \theta^k \in W^*$, then

$$
\begin{aligned}
&(L^*(\theta^1 \wedge \cdots \wedge \theta^k))(v_1, \ldots, v_k) \\
&= \theta^1 \wedge \cdots \wedge \theta^k)(L(v_1), \ldots, L(v_k)) \\
&= \sum_{\sigma \in S_k} \epsilon(\sigma) \langle \theta^{\sigma(1)}, L(v_1) \rangle \cdots \langle \theta^{\sigma(k)}, L(v_k) \rangle \\
&= \sum_{\sigma \in S_k} \epsilon(\sigma) \langle L^* \theta^{\sigma(1)}, v_1 \rangle \cdots \langle L^* \theta^{\sigma(k)}, v_k \rangle \\
&= ((L^*\theta^1) \wedge \cdots \wedge (L^*\theta^k))(v_1, \ldots, v_k)
\end{aligned}
$$

- Therefore,

$$
L^*(\theta^1 \wedge \cdots \wedge \theta^k) = (L^*\theta^1) \wedge \cdots \wedge (L^*\theta^k)
$$

Wedge Product of 1-Forms

- Let $\theta^1, \ldots, \theta^k$ be 1-forms on O
- For each $x \in O$,

$$\theta^1(x), \ldots, \theta^k(x) \in T_x^* O$$

- Therefore,

$$\theta^1(x) \wedge \cdots \wedge \theta^k(x) \in \Lambda_x^k O$$

- $\theta^1 \wedge \cdots \wedge \theta^k$ is the differential k-form such that for each $x \in O$,

$$(\theta^1 \wedge \cdots \wedge \theta^k)(x) = \theta^1(x) \wedge \cdots \wedge \theta^k(x) \in \Lambda_x^k O$$

Pullback of Wedge Product of 1-Forms

- Let $F : O \to P$ be a smooth map
- If $\phi^1, \ldots, \phi^k$ are differential forms on P, then

$$F^*(\phi^1 \wedge \cdots \phi^k) = (F^*\phi^1) \wedge \cdots \wedge (F^*\phi^k)$$

Differential Forms in Coordinates (Part 1)

- Let $(x^1, \ldots, x^m)$ denote coordinates on O
- For each $x \in O$, $(dx^1, \ldots, dx^m)$ is a basis of $T_x^* O$
- For each $x \in O$,

$$(dx^i \wedge dx^j : \ 1 \leq i < j \leq m)$$

is a basis of $\Lambda_x^2 O$ and therefore

$$\dim(\Lambda_x^2 O) = \frac{1}{2} m(m-1)$$

- For each $x \in O$ and $1 \leq k \leq m$,

$$(dx^{i_1} \wedge dx^{i_k} : \ 1 \leq i_1 < i_2 < \cdots < i_k \leq m)$$

is a basis of $\Lambda_x^k O$ and therefore

$$\dim(\Lambda_x^k O) = \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k(k-1)\cdots(2)1} = \frac{m!}{(m-k)!k!}$$

Differential Forms in Coordinates (Part 2)

- Any differential k-form on $O \subset \mathbb{R}^m$ can be written as

$$\Theta = \frac{1}{k!} \sum_{1 \le i_1, \ldots, i_k \le m} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

  where each $a_{i_1 \cdots i_k}$ is a smooth scalar function on O and for any $\sigma \in S_k$,

$$a_{\sigma(i_1) \cdots \sigma(i_k)} = \epsilon(\sigma) a_{i_1 \cdots i_k}$$

- Any 2-form on O can be written as

$$\begin{aligned}
\Theta &= \frac{1}{2} \sum_{1 \le i_1, i_2 \le m} a_{i_1 i_2} dx^{i_1} \wedge dx^{i_2} \\
&= \frac{1}{2} \sum_{1 \le i_1 < i_2 \le m} (a_{i_1 i_2} dx^{i_1} \wedge dx^{i_2} + a_{i_2 i_1} dx^{i_2} \wedge dx^{i_1} \\
&= \sum_{1 \le i_1 < i_2 \le m} a_{i_1 i_2} dx^{i_1} \wedge dx^{i_2}
\end{aligned}$$

Pullback of Differential Form in Coordinates

- Given any differential k-form on $P \subset \mathbb{R}^n$

$$\Omega = \frac{1}{k!} \sum_{1 \le p_1, \ldots, p_k \le n} b_{p_1 \cdots p_k} dy^{p_1} \wedge \cdots \wedge dy^{p_k}$$

and smooth map $F : O \to P$, the pullback of Ω by F is

$$\begin{aligned}
F^*\Omega &= \frac{1}{k!} \sum_{1 \le p_1, \ldots, p_k \le n} b_{p_1 \cdots p_k} dy^{p_1} \wedge \cdots \wedge dy^{p_k} \\
&= \frac{1}{k!} \sum_{1 \le p_1, \ldots, p_k \le n} b_{p_1 \cdots p_k} dy^{p_1} \wedge \cdots \wedge dy^{p_k} \\
&= \frac{1}{k!} \sum_{1 \le p_1, \ldots, p_k \le n} b_{p_1 \cdots p_k} \left(\frac{y^{p_1}}{\partial x^{j_1}} \, dx^{j_1} \right) \wedge \cdots \wedge \left(\frac{y^{p_k}}{\partial x^{j_k}} \, dx^{j_k} \right) \\
&= \frac{1}{k!} \sum_{\substack{1 \le p_1, \ldots, p_k \le n \\ 1 \le j_1, \ldots, j_k \le m}} b_{p_1 \cdots p_k} \frac{y^{p_1}}{\partial x^{j_1}} \cdots \frac{y^{p_k}}{\partial x^{j_k}} \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}
\end{aligned}$$

Example: Polar Coordinates (Part 1)

- Let
$$O = \{(r,\theta)\ :\ r>0 \text{ and } -\pi<\theta<\pi\} \subset \mathbb{R}^2$$
$$P = \{(x,y) \in \mathbb{R}^2\}$$
- Let $F : O \to P$ be given by
$$F(r,\theta) = (r\cos\theta, r\sin\theta),$$
i.e.,
$$x(r,\theta) = r\cos\theta \text{ and } y(r,\theta) = r\sin\theta$$
- Consider the 2-form
$$\Theta = dx \wedge dy$$
- Then
$$\begin{aligned} F^*dx &= d(r\cos\theta) \\ &= dr\cos\theta - r\sin\theta\, d\theta \\ F^*dy &= d(r\sin\theta) \\ &= dr\sin\theta + r\cos\theta\, d\theta \end{aligned}$$

Example: Polar Coordinates (Part 2)

- If
$$\Theta = dx \wedge dy,$$
then
$$\begin{aligned} F^*\Theta &= F^*(dx \wedge dy) \\ &= (F^*dx) \wedge (F^*dy) \\ &= (dr\cos\theta - r\sin\theta\, d\theta) \wedge (dr\sin\theta + r\cos\theta\, d\theta) \\ &= r\, dr \wedge d\theta \end{aligned}$$

Example: Spherical Coordinates (Part 1)

- Let

$$O = \{(\rho, \phi, \theta) : \ \rho > 0, \ 0 < \phi < \pi, \ -\pi < \theta < \pi\} \subset \mathbb{R}^3$$
$$P = \{(x, y, z) \in \mathbb{R}^3\}$$

- Let $F : O \to P$ be given by

$$F(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

- Therefore,

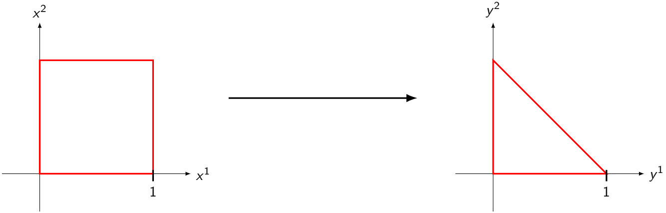
$$\begin{aligned}
F^* dx &= d(\rho \sin\phi \cos\theta) \\
&= d\rho \sin\phi \cos\theta + d\phi(\rho \cos\phi \cos\theta - d\theta(\rho \sin\phi \sin\theta) \\
F^* dy &= d(\rho \sin\phi \sin\theta) \\
&= d\rho(\sin\phi \sin\theta) + d\phi(\rho \cos\phi \sin\theta) + d\theta(\rho \sin\phi \cos\theta) \\
F^* dz &= d(\rho \cos\phi) \\
&= d\rho(\cos\phi) - d\phi(\rho \sin\phi)
\end{aligned}$$

Example: Spherical Coordinates (Part 2)

- If $\Theta = dy \wedge dz$, then

$$
\begin{aligned}
F^*\Theta &= F^*(dy \wedge dz) \\
&= F^*dy \wedge F^*dz \\
&= (d\rho \sin\phi \sin\theta + \rho(\cos\phi\, d\phi)\sin\theta + \rho \sin\phi \cos\theta\, d\theta) \\
&\quad \wedge (d\rho \cos\phi - \rho \sin\phi\, d\phi) \\
&= (-\rho(\sin\phi)^2 \sin\theta - \rho(\cos\phi)^2 \sin\theta)\, d\rho \wedge d\phi \\
&\quad - \rho \sin\phi \cos\phi \cos\theta\, d\rho \wedge d\theta + \rho^2(\sin\phi)^2 \cos\theta\, d\phi \wedge d\theta \\
&= -\rho \sin\theta\, d\rho \wedge d\phi \\
&\quad - \rho \sin\phi \cos\phi \cos\theta\, d\rho \wedge d\theta + \rho^2(\sin\phi)^2 \cos\theta\, d\phi \wedge d\theta
\end{aligned}
$$

Example: Parameterization of Triangle



- $(y^1, y^2) = ((1 - x^2)x^1, x^2)$
- The differentials are

$$dy^1 = (1 - x^2)\, dx^1 - x^1\, dx^2$$
$$dy^2 = dx^2$$

- Therefore, the pullback of $dy^1 \wedge dy^2$ is

$$dy^1 \wedge dy^2 = ((1 - x^2)\, dx^1 - x^1\, dx^2) \wedge dx^2 = (1 - x^2) dx^1 \wedge dx^2$$