MATH-GA1002 Multivariable Analysis

Implicit Function Theorem
Normal Form for Submersion
Normal Form for Immersion
Atlas of Coordinate Maps
Definition of Manifold

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 6, 2024

Chain Rule for Maps

- Given an open $O \subset \mathbb{R}^n$, a C^1 map

$$F : O \to \mathbb{R}^m,$$

an open $U \subset \mathbb{R}^m$, $F(O) \subset U$, and a C^1 map

$$G : U \to \mathbb{R}^k,$$

the chain rule states that

$$\partial(G \circ F)(x) = (\partial G(F(x))) \circ (\partial F(x))$$

- First, recall that given any $x \in O$, and $v \in \mathbb{R}^n$, then for any C^1 curve

$$c : I \to O, \text{ where } c(0) = x \text{ and } c'(0) = v,$$

it follows that

$$\partial F(x)(v) = \left.\frac{d}{dt}\right|_{t=0} F(c(t))$$

Proof of Chain Rule for Maps

- Given $x \in O$ and $v \in \mathbb{R}^n$, let

$$\begin{aligned} c : I &\to U \\ t &\mapsto F(x + tv) \end{aligned}$$

Observe that

$$c(0) = F(x) \text{ and } c'(0) = \partial F(x)(v)$$

- Then the differential of $G \circ F$ at x is

$$\begin{aligned} \partial(G \circ F)(x)(v) &= \left.\frac{d}{dt}\right|_{t=0} G(F(x + tv)) \\ &= \left.\frac{d}{dt}\right|_{t=0} G(c(t)) \\ &= \partial G(F(x))(c'(0)) \\ &= \partial G(F(x)(\partial F(x)(v)) \end{aligned}$$

- Therefore,

$$\partial(G \circ F)(x) = (\partial G(F(x))) \circ (\partial F(x))$$

Linear Implicit Function Theorem

- Let $m, n > 0$ and $M : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be a linear map of the form

$$M = \left[\begin{array}{c|c} A_{m \times m} & B_{m \times n} \end{array} \right],$$

where A is invertible

- Then there exists a unique linear map

$$N : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

such that

$$M \begin{bmatrix} v \\ w \end{bmatrix} = u \iff \begin{bmatrix} v \\ w \end{bmatrix} = N \begin{bmatrix} u \\ w \end{bmatrix} \tag{1}$$

- Moreover, N is a linear isomorphism

Proof of Linear Implicit Function Theorem

- For any $(v, w) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$M \begin{bmatrix} v \\ w \end{bmatrix} = Av + Bw.$$

- Therefore, for each $u \in \mathbb{R}^m$,

$$\begin{aligned} M \begin{bmatrix} v \\ w \end{bmatrix} = u &\iff Av + Bw = u \\ &\iff v = A^{-1}(u - Bw) \\ &\iff \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \end{aligned}$$

- Therefore,

$$N = \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}$$

Another Proof of Linear Implicit Function Theorem

- Let $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ be the linear map

$$L \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} Av + Bw \\ w \end{bmatrix},$$

i.e.,

$$L = \left[\begin{array}{c|c} A & B \\ \hline 0_{n\times m} & I_{n\times n} \end{array}\right]$$

- Observe that L is invertible and

$$L^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}$$

- Then

$$M \begin{bmatrix} v \\ w \end{bmatrix} = u \iff L \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} u \\ w \end{bmatrix}$$
$$\iff L^{-1} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$$

Implicit Function Theorem

- Let $m, n > 0$, O be an open neighborhood of $0 \in \mathbb{R}^{n+m}$ and
$$f : O \to \mathbb{R}^m$$
be a C^1 map such that $f(0)$ and
$$\partial f(0) : \mathbb{R}^{m+n} \to \mathbb{R}^m$$
is a matrix of the form
$$\partial f(0) = \left[\ A_{m\times m} \ \middle|\ B_{m\times n}\ \right],$$
where A is invertible
- Then there exists an open neighborhood N of $0 \in \mathbb{R}^{m+n}$ and a unique C^1 map
$$\phi : N \to O$$
such that for any $(z, y) \in N$,
$$(x, y) \in \phi(N) \text{ and } f(x, y) = z \iff (x, y) = \phi(z, y)$$
- Moreover, ϕ is a diffeomorphism

Proof of Implicit Function Theorem (Part 1)

- Let $F : O \to \mathbb{R}^{n+m}$ be given by

$$F(x,y) = (f(x,y), y)$$

- The differential of F at $(0,0)$ is a linear map

$$\begin{aligned} \partial F(0,0) : \mathbb{R}^{n+m} &\to \mathbb{R}^{n+m} \\ \begin{bmatrix} v \\ w \end{bmatrix} &\mapsto \begin{bmatrix} \partial f(0,0)(v,w) \\ w \end{bmatrix} \\ &= L \begin{bmatrix} v \\ w \end{bmatrix}, \end{aligned}$$

  where

$$L = \begin{bmatrix} A & B \\ 0_{n\times m} & I_{n\times n} \end{bmatrix},$$

- Since A is invertible, so is L

Proof of Implicit Function Theorem (Part 2)

- Since $L = \partial F(0,0)$ is invertible, it follows by the inverse function theorem that there exist an open neighborhood N of $0 \in \mathbb{R}^{n+m}$ and a unique C^1 map

$$F^{-1} : N \to O$$

  such that $F(F^{-1}(z,y)) = (z,y)$ for any $(z,y) \in N$
- If $F^{-1}(z,y) = (\phi_1(z,y), \psi_2(z,y))$, then

$$(z,y) = F(F^{-1}(z,y)) = F(\phi_1(z,y), \phi_2(z,y)) = (f(\phi_1(z,y), \phi_2(z,y$$

  which holds if and only if $\phi_2(z,y) = y$ and $f(\phi_1(z,y), y) = z$
- It follows that for any $(z,y) \in N$,

$$(x,y) \in F^{-1}(N) \text{ and } F(x,y) = (z,y) \iff F(x,y) = (z,y)$$

Normal Form for Surjective Linear Map

- Let $\dim(V) = m + n$, $\dim(W) = m$, and $L : V \to W$ be a linear maps with rank m
- **Fact:** There exists linear isomorphisms $A : W \to \mathbb{R}^m$ and $B : \mathbb{R}^{n+m} \to V$ such that the linear map

$$M = A \circ L \circ B : \mathbb{R}^{m+n} \to \mathbb{R}^m$$

is the matrix

$$M = \left[\; I_{m\times m} \;\middle|\; 0_{m\times n} \;\right],$$

i.e., for any $(x', x'') \in \mathbb{R}^m \times \mathbb{R}^n$,

$$M \begin{bmatrix} x' \\ x'' \end{bmatrix} = x'$$

Proof

- Since the rank of L is m, $\dim(\ker(L)) = n$
- Let $(e_{m+1}, \ldots, e_{m+n})$ be a basis of $\ker(L)$
- Extend this to a basis $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n})$ of V
- For each $1 \leq j \leq m$, let $f_j = L(e_j)$
- $(f_1, \ldots, f_m)$ is linearly independent and therefore a basis of W
- Therefore, for any $1 \leq a \leq m+n$,

$$L(e_a) = \begin{cases} f_a & \text{if } 1 \leq a \leq m \\ 0 & \text{if } m+1 \leq a \leq m+n \end{cases}$$

- Let $(\epsilon_1, \ldots, \epsilon_N)$ be the standard basis of $\mathbb{R}^N$
- Let $A : W \to \mathbb{R}^m,\ B : \mathbb{R}^{n+m} \to V$ be linear maps given by

$$A(f_j) = \epsilon_j,\ \forall 1 \leq j \leq m$$
$$B(\epsilon_a) = e_a,\ \forall 1 \leq a \leq m+n$$

- Then $M = A \circ L \circ B : \mathbb{R}^{m+n} \to \mathbb{R}^m$ satisfies

$$M(\epsilon_a) = \begin{cases} \epsilon_a & \text{if } 1 \leq a \leq m \\ 0 & \text{if } m+1 \leq a \leq m+n \end{cases}$$

Normal Form for Submersion

- Let O be an open neighborhood of $0 \in \mathbb{R}^{m+n}$ and

$$\Phi : O \to \mathbb{R}^m$$

be a C^k submersion such that $\Phi(0) = 0$

- There exists a neighborhood $U \subset O$ of 0, and a diffeomorphisms

$$R : \Phi(U) \to \mathbb{R}^m$$
$$S : S^{-1}(U) \to U$$

such that the map

$$\Psi = R \circ \Phi \circ S : S^{-1}(U) \to \mathbb{R}^m$$

is given by

$$\Psi(x', x'') = x', \ \forall (x', x'') \in S^{-1}(U)$$

Proof of Normal Form for Submersion

- Since
$$L = \partial\Phi(0) : \mathbb{R}^{m+n} \to \mathbb{R}^m$$
has rank m, there exist linear isomorphisms
$$A : \mathbb{R}^m \to \mathbb{R}^m \text{ and } B : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$$
such that
$$A \circ L \circ B = \begin{bmatrix} I_{m\times m} & 0_{m\times n} \end{bmatrix}$$
- Therefore, if
$$\Psi = A \circ \Phi \circ B : B^{-1}(O) \to \mathbb{R}^m,$$
then the differential of Ψ at (0) is
$$\partial\Psi(0) = \begin{bmatrix} I_{m\times m} & 0_{m\times n} \end{bmatrix}$$
- The theorem now follows by the implicit function theorem

Normal Form for Injective Linear Map

- Let $\dim(V) = m$, $\dim(W) = m + n$, and $L : V \to W$ be a linear maps with rank m
- **Fact:** There exists linear isomorphisms $A : W \to \mathbb{R}^{m+n}$ and $B : \mathbb{R}^m \to V$ such that the linear map

  $$M = A \circ L \circ B : \mathbb{R}^m \to \mathbb{R}^{m+n}$$

  is the matrix

  $$M = \left[\begin{array}{c} I_{m\times m} \\ \hline 0_{n\times m} \end{array}\right],$$

  i.e., for all $x' \in \mathbb{R}^m$,

  $$Mx' = \begin{bmatrix} x' \\ 0'' \end{bmatrix}$$

Proof

- Let $(e_{m+1}, \ldots, e_m)$ be a basis of V
- For each $1 \leq j \leq m$, let $f_j = L(e_j)$
- Since $\ker(L) = \{0\}$, $(f_1, \ldots, f_m)$ is linearly independent
- Extend to a basis $(f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n})$
- Let $A : W \to \mathbb{R}^{m+n}$, $B : \mathbb{R}^{n+m} \to V$ be linear maps given by

$$A(f_a) = \epsilon_a, \ \forall 1 \leq a \leq m + n$$
$$B(\epsilon_j) = e_j, \ \forall 1 \leq j \leq m$$

- Then $M = A \circ L \circ B : \mathbb{R}^{m+n} \to \mathbb{R}^m$ satisfies

$$M(\epsilon_j) = \epsilon_j, \ \forall 1 \leq j \leq m$$

Normal Form for Immersion

- Let O' be an open neighborhood of $0 \in \mathbb{R}^m$ and $\Phi : O' \to \mathbb{R}^{m+n}$ be a C^k immersion such that $Phi(0) = 0$
- Then there exists a neighborhood $U' \subset O'$ of 0, a neighborhood $U \subset \mathbb{R}^{m+n}$ of 0, and diffeomorphisms

$$R : U \to \mathbb{R}^{m+n}$$
$$S : S^{-1}(U') \to U'$$

such that $\Phi(U') \subset U$ and the map

$$\Psi = R \circ \Phi \circ S : S^{-1}(U') \to \mathbb{R}^{m+n}$$

is given by

$$\Phi(x') = (x', 0) \in \mathbb{R}^m \times \mathbb{R}^n$$

Proof of Normal Form for Immersion (Part 1)

- Since
$$L = \partial\Phi(0) : \mathbb{R}^m \to \mathbb{R}^{m+n}$$
has rank m, there exist linear isomorphisms
$$A : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \text{ and } B : \mathbb{R}^m \to \mathbb{R}^m$$
such that
$$A \circ L \circ B = \begin{bmatrix} I_{m\times m} \\ 0_{n\times m} \end{bmatrix}$$
- Therefore, the map $\Psi = A \circ \Phi \circ B$ has differential at 0 equal to
$$\partial\Psi(0) = \begin{bmatrix} I_{m\times m} \\ 0_{n\times m} \end{bmatrix}$$

Proof of Normal Form for Immersion (Part 2)

- Now define the map

$$\begin{aligned} F : B^{-1}(O) \times \mathbb{R}^n &\to \mathbb{R}^m \times \mathbb{R}^n \\ (x, y) &\mapsto (\Psi(x, y), y) \end{aligned}$$

- The differential of F at $(0, 0) \in B^{-1}(O)$ is

$$\begin{aligned} \partial F(0, 0) : \mathbb{R}^m \times \mathbb{R}^n &\to \mathbb{R}^m \times \mathbb{R}^n \\ \begin{bmatrix} v \\ w \end{bmatrix} &\mapsto \begin{bmatrix} I_{m \times m} & 0_{m \times n} \\ 0_{n \times m} & I_{n \times n} \end{bmatrix} = I_{(m+n) \times (m+n)} \end{aligned}$$

- The theorem now follows by the inverse function theorem

Linear Maps and Bases of Vector Spaces

- If V is an n-dimensional vector space, then any basis $(b_1, \ldots, b_n)$ of V defines a linear isomorphism

$$\begin{aligned} \mathbb{R}^n &\to V \\ (r^1, \ldots, r^n) &\mapsto r^1 b_1 + \cdots + r^n b_n = r^k b_k \end{aligned}$$

- Conversely, any linear isomorphism

$$L : \mathbb{R}^n \to V$$

  defines a basis $(b_1, \ldots, b_n)$ where

$$b_k = L(e_k)$$

- For any linear isomorphisms $L_1, L_2 : \mathbb{R}^n \to V$, $L_2^{-1} \circ L_1 : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism
- The set of all linear isomorphisms $L : \mathbb{R}^n \to V$ is a **linear atlas** of V

Linear Atlas of a Set

- An n-**dimensional linear atlas** of a nonempty set S is a nonempty collection $\mathcal{A}$ of bijective maps $\Phi : \mathbb{R}^n \to S$ such that for any $\Phi_1, \Phi_2 \in \mathcal{A}$, the map

$$\Phi_2^{-1} \circ \Phi_1 : \mathbb{R}^n \to \mathbb{R}^n$$

  is a linear isomorphism
- A linear atlas on S implies a unique vector space structure on S such that the maps in the atlas are linear
- An atlas can consist of only one map
- Given an atlas $\mathcal{A}$, there is maximal atlas that contains all possible linear maps $\Phi : \mathbb{R}^n \to S$

Linear Maps

- If S has an n-dimensional linear atlas $\mathcal{S}$ and T has an m-dimensional atlas $\mathcal{T}$, then a map

$$L : S \to T$$

is linear if and only if for any $\Phi \in \mathcal{S}$ and $\Psi \in \mathcal{T}$, the map

$$\Psi \circ \Phi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$$

is linear

Nonlinear Atlas of a Set

- An n-**dimensional** C^k **atlas** of a nonempty set S is a nonempty collection $\mathcal{A}$ of bijective maps $\Phi : O \to S$, where $O \subset \mathbb{R}^n$ is open, such that for any maps

  $$\Phi_1 : O_1 \to S \text{ and } \Phi_2 : O_2 \to S$$

  in $\mathcal{A}$, the map

  $$\Phi_2^{-1} \circ \Phi_1 : O_1 \to O_2$$

  is a C^k diffeomorphism
- A bijective map $\Psi : U \to S$, where $U \subset \mathbb{R}^n$ is open is **compatible** with an n-dimensional C^k local atlas $\mathcal{A}$ if for any $\Phi : O \to S$ in the atlas, the map

  $$\Phi^{-1} \circ \Psi : U \to O$$

  is a C^k diffeomorphism
- Given an atlas $\mathcal{A}$, there is maximal atlas that contains all maps $\Phi : \mathbb{R}^n \to S$ that are compatible with $\mathcal{A}$

C^k Manifolds

- A set S with a C^k atlas is an example of a C^k **manifold**
- Any open $O \subset \mathbb{R}^n$ is an n-dimensional C^k manifold
- A C^k manifold is an abstract space that is a nonlinear analogue of an abstract vector space
- Any map $\Phi : U \to S$ in the atlas $\mathcal{S}$ is called a **coordinate map**
- The inverse map $\Phi^{-1} : S \to U$ will also be called a coordinate map
- Below, we will restrict to manifolds with atlases and coordinate maps of this form

C^k Maps

- If S is an n-dimensional C^k manifold with atlas $\mathcal{S}$ and T is an m-dimensional C^k manifold with atlas $\mathcal{T}$, then a map

$$F : S \to T$$

is C^k if and only if for any maps

$$\Phi : O \to S \text{ in } \mathcal{S} \text{ and } \Psi : U \to S \text{ in } \mathcal{T},$$

the map

$$\Psi \circ \Phi^{-1} : O \to U$$

is C^k