MATH-GA1002 Multivariable Analysis

Differential of a Function
Smooth Functions
Directional Derivatives of a Function
Derivations

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Differential of a Function With Respect to Coordinates

- Recall that given a C^1 function $f : O \to \mathbb{R}$, its differential at $x \in O$ is the linear function
$$\begin{aligned} df(x) : \mathbb{R}^n &\to \mathbb{R} \\ v &\mapsto D_v f(x) \end{aligned}$$
- In particular, for each $1 \leq k \leq n$,
$$\begin{aligned} \langle df(x), e_k \rangle &= D_{e_k} f(x) \\ &= \partial_k f(x) \end{aligned}$$
- Since $(dx^1, \dots, dx^n)$ is the dual basis to $(e_1, \dots, e_n)$, it follows that
$$df(x) = \partial_k f(x)\, dx^k$$
- Therefore, for any $v = e_k v^k \in \mathbb{R}^n$,
$$\begin{aligned} \langle df(x), v \rangle &= \langle \partial_j f(x)\, dx^j, e_k v^k \rangle \\ &= \partial_j f(x) v^k \langle dx^j, e_k \rangle \\ &= \partial_k f(x) v^k \end{aligned}$$

Smooth Functions

- Let $O \subset \mathbb{R}^n$ be open and consider a function $f : O \to \mathbb{R}$
- f is C^0 on O if it is continuous
- f is C^1 on O if it is continuous and its partial derivatives are continuous
- Lemma: *If f is C^1, then it is differentiable*
- f is C^k if f and its partial derivatives up to order k are continuous
- f is C^∞ or **smooth** if f and its partial derivatives of all orders are continuous
- The space of all C^k functions with domain O is denoted $C^k(O)$

Properties of Directional Derivatives

- Given $x_0 \in O$ and $v \in \mathbb{R}^n$, the directional derivative D_v is an operator

$$D_v : C^1(O) \to \mathbb{R}$$
$$f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(x + tv)$$

- For any $f_1, f_2 \in C^1(O)$ and $a^1, a^2 \in \mathbb{R}$,
  - Constant factor and sum rules:

$$(D_v(a^1 f_1 + a^2 f_2))(x) = a^1(D_v f^1)(x) + a^2(D_v f^2)(x)$$

  - Product rule:

$$(D_v(f_1 f_2))(x) = f_2(x)(D_v f_1)(x) + f_1(x)(D_v f_2)(x)$$

Derivations

- Given an open $O \subset \mathbb{R}^n$, a map

$$D : C^\infty(O) \to \mathbb{R}$$

  is a **derivation** at $x \in O$ if the following holds:
- For any $f_1, f_2 \in C^\infty(O)$ and $a^1, a^2 \in \mathbb{R}$,
  - Constant factor and sum rules:

$$D(a^1 f_1 + a^2 f_2) = a^1 Df^1 + a^2 Df^2$$

  - Product rule:

$$D(f_1 f_2) = f_2(x) Df_1 + f_1(x) Df_2$$

- A directional derivative is a derivation

Derivation of Constant Function is Zero

- Let D be a derivation at $x_0 \in O$
- For any $c \in \mathbb{R}$ and $f \in C^\infty(O)$,

$$\begin{aligned} D(cf) &= f(x_0)D(c) + cD(f), \text{ by product rule} \\ &= f(x_0)D(c) + D(cf), \text{ by constant factor fule} \end{aligned}$$

- Therefore $D(c) = 0$

A Derivation is a Directional Derivative (Part 1)

- By the Fundamental Theorem of Calculus,

$$
\begin{aligned}
f(x) &= f(x_0) + \int_{t=0}^{t=1} \frac{d}{dt} f(x_0 + t(x - x_0))\, dt \\
&= f(x_0) + \int_{t=0}^{t=1} (x^i - x_0^i)\partial_i f(x_0 + t(x - x_0))\, dt \\
&= f(x_0) + (x^i - x_0^i) \int_{t=0}^{t=1} \partial_i f(x_0 + t(x - x_0))\, dt \\
&= f(x_0) + \phi^i(x) b_i(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi^i(x) &= x^i - x_0^i \\
b_i(x) &= \int_{t=0}^{t=1} \partial_i f(x_0 + t(x - x_0))\, dt
\end{aligned}
$$

A Derivation is a Directional Derivative (Part 2)

- Therefore, if $v^i = D(\phi^i)$, then

$$\begin{aligned} D(f) &= D(f(x_0) + \phi^i b_i) \\ &= b_i(x_0)D(\phi^i) + \phi^i(x_0)D(b_i) \\ &= v^i \partial_i f(x_0) \\ &= D_v f(x_0) \end{aligned}$$

Fundamental Examples of Smooth Functions

- Constant function: For any $c \in \mathbb{R}$, the function

$$f(x) = c, \ \forall x \in \mathbb{R}^n$$

  - $df = 0$

- Coordinate functions: For each $i \in \{1, \ldots, n\}$, the function

$$x^i : \mathbb{R}^n \to \mathbb{R}$$
$$(a^1, \ldots, a^n) \mapsto a^i$$

  - $dx^i = \epsilon^i$

Partial Derivatives Commute

- If $f : O \to \mathbb{R}$ is C^2, then

$$(\partial_i(\partial_j f))(x) = (\partial_j(\partial_i f))(x)$$

- We shall denote

$$\partial^2_{ij} f = \partial_i(\partial_j f) = \partial_j(\partial_i f)$$