Problemset 10

Q1. \(X\) and \(Y\) are Banach spaces. \(T_n\) is a sequence of bounded operators \(X \to Y\) such that \(\sup_n \|T_n\| \leq C\) and
\[
\lim_{n \to \infty} T_n x = T x
\]
eexists for \(x \in D\), a dense subspace of \(X\). Show that
\[
\lim_{n \to \infty} T_n x = T x
\]
eexists for all \(x \in X\) and \(\|T\| \leq C\).

Q2. If \(X\) and \(Y\) are Hilbert spaces and \(T\) is an isometry between dense subspaces \(D_1 \subset X\) and \(D_2 \subset Y\), then \(T\) extends as an isometry between \(X\) and \(Y\).

Q3. The space \(S\) of functions on \(R\) consists of smooth functions that satisfy for nonnegative integers \(n\) and \(r\)
\[
\left| \frac{d^n f(x)}{dx^n} \right| \leq C_{r,n} (1 + x^2)^{-r}
\]
for some constants \(C_{r,n}\). Show that \(f \in S\) if and only if it Fourier transform
\[
(\hat{f})(x) = \int e^{ixy} f(y) dy \in S
\]