Problem 1. If $\mu$ is a measure with $\mu(R) = 1$ on the Borel $\sigma$–field $\mathcal{B}(R)$, its distribution function is defined by

$$F(x) = \mu([(-\infty, x]]\right) = \mu[\{y : -\infty < y \leq x\}].$$

Show that $F(x)$ is nondecreasing, continuous from the right and

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$$

Conversely show that given such a function $F$ there is a unique $\mu$ such that

$$F(x) = \mu([(-\infty, x]]).$$

Problem 2. Find a sequence $f_n(x)$ on $[0, 1]$ such that $f_n(\cdot) \to 0$ in measure with respect to the Lebesgue measure but $\lim \sup_n f_n(x) = 1$ for every $x$.

Problem 3.

Consider $f_n(x) = n^p x^n (1 - x)$ on $[0, 1]$. $\lim_{n \to \infty} f_n(x) = 0$ for $x \in [0, 1]$. Determine the values of $p$ for which $\int_{[0,1]} f_n(x)dx \to 0$. Are these the same as those for which $\sup_n f_n(x)$ is integrable on $[0, 1]$?