Weak Topology. A weak open set around \( x \in \mathcal{X} \) is given by

\[
N(x : n, \Lambda_1, \ldots, \Lambda_n) = \{ y : |\Lambda_i(x) - \Lambda_i(y)| \leq \epsilon, \forall 1 \leq i \leq n \}
\]

for a finite collection of linear functionals \( \{\Lambda_i\} \) in the dual \( \mathcal{X}^* \) of \( \mathcal{X} \). It is not metrizable! There is no countable basis at 0 unless \( \mathcal{X}^* \) and therefore \( \mathcal{X} \) is finite dimensional. But if \( \mathcal{X}^* \) is separable then the unit ball, with weak topology is metrizable and is in fact compact. With a countable dense subset \( \{\Lambda_i\} \) of \( \mathcal{X}^* \)

\[
d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |\Lambda_i(x) - \Lambda_i(y)|
\]

will do it. We can try the weak topology on the dual \( \mathcal{X}^* \). Either we can try the linear functionals \( <\Lambda, x> = \Lambda(x) \) as linear in \( x \) for fixed \( \Lambda \) or linear in \( \Lambda \) for fixed \( x \). So \( \mathcal{X}^* \) has two weak topologies using linear functionals \( x(\Lambda) \) from \( \mathcal{X} \) or \( x^{**}(\Lambda) \) from \( \mathcal{X}^{**} \). Since \( \mathcal{X} \subset \mathcal{X}^{**} \) one is weaker than the other. The weak topology on \( \mathcal{X}^* \) can come from considering either \( \mathcal{X} \) or \( \mathcal{X}^{**} \). One hardly ever chooses \( \mathcal{X}^{**} \). In many examples like \( L_p \) spaces with \( 1 < p < \infty \), \( \mathcal{X} = \mathcal{X}^{**} \). Such spaces are called reflexive Banach spaces.

Weak compactness. The Unit Ball in \( L_p \) for \( 1 < p < \infty \) is compact in the weak topology. \( L_1 \) is different. We have functions \( f_n(x) \) such that \( \int |f_n(x)|d\mu \leq 1 \) may not have a weak limit. For example \( f_n(x) = n1_{[0, \frac{1}{n}]} \) in \( L_1[0, 1] \) with Lebesgue measure. The weak limit wants to be the \( \delta \)-function at 0. Need uniform integrability.

A finite dimensional subspace of a Banach space is closed. Let \( S = \{a_1x_1 + \cdots + a_dx_d\} \) for some fixed linearly independent \( x_1, \ldots, x_d \in \mathcal{X} \) and \( a_1, \ldots, a_d \in \mathbb{R}^d \). Let \( S \ni x_n = a_1^n x_1 + \cdots + a_d^n x_d \) and \( x_n \to x \). If \( \tau_n = \sup_{n,j} |a_j^n| \) is bounded then we can choose subsequences so that \( a_j^n \to a_j \) and \( x = a_1x_1 + \cdots + a_dx_d \in S \). If \( \tau_n \) is unbounded we can divide both sides of

\[
x_n = a_1^n x_1 + \cdots + a_d^n x_d
\]

by \( \tau_n \). The left side will \( \to 0 \). The terms on the right \( \frac{a_j^n}{\tau_n} \) will be bounded and if we take a limit of subsequence \( a_j^n \to a_j \) and at least one \( a_j \) will be such that \( |a_j| = 1 \).

\[
\sum a_j x_j = 0
\]

contradicting linear independence.

The unit ball \( \|x\| \leq 1 \) can not be compact if \( \mathcal{X} \) is not finite dimensional. Let \( \mathcal{X} \) be infinite dimensional. Given any \( \alpha < 1 \) there is a sequence \( x_n \) such that \( \|x_n\| = 1 \) for all \( n \) and \( \|x_i - x_j\| \geq \alpha \) for all \( i \neq j \). It is enough to show that given a closed subspace \( S \subset \mathcal{X} \), \( S \neq \mathcal{X} \), and \( \alpha < 1 \), there is a \( y \in \mathcal{X} \) such that \( \|y\| = 1 \) and \( \inf_{x \in S} \|y - x\| \geq \alpha \).

Take \( y \notin S \) with \( \|y\| = 1 \). Since \( S \) is closed \( \inf_{x \in S} \|y - x\| = \theta > 0 \) For any \( \epsilon > 0 \) can find \( x_1 \in S \) such that \( \|y - x_1\| \leq \theta + \epsilon \). Let \( y_1 = \frac{(y - x_1)}{\|y - x_1\|} \). Then \( \|y_1\| = 1 \). Since \( S \) is a subspace for \( \epsilon \) small

\[
d(y_1, S) = d\left(\frac{y}{\|y - x_1\|}, S\right) = \frac{1}{\|y - x_1\|} d(y, S) \geq \frac{\theta}{\theta + \epsilon} \geq \alpha
\]

An operator $T$ from $\mathcal{X}$ to $\mathcal{Y}$ is compact or completely continuous if the image of the unit ball of $\mathcal{X}$ is a compact set in $\mathcal{Y}$. $T_1, T_2$ compact implies $T_1 + T_2$ is compact. $T_1 : \mathcal{X} \rightarrow \mathcal{Y}$ $T_2 : \mathcal{Y} \rightarrow \mathcal{Z}$. If one of them is bounded and the other is compact the composition is compact. A bounded operator maps compact sets into compact sets. $T_n$ compact for each $n, ||T_n - T|| \rightarrow 0$ implies $T$ is compact. Let $x_k \in \mathcal{X}$ satisfy $||x_k|| \leq 1$. Since $T_n$ is compact there is a subsequence such that $T_n x_k \rightarrow y_n$ as $k \rightarrow \infty$. We can diagonalize and assume this happens for all $n$. We want to show that $T x_k$ has a limit.

$$
|| Tx_i - Tx_j || \leq || T_n x_i - T_n x_j || + || T_n - T || || x_i - x_j ||
$$

$$
\limsup_{i,j \rightarrow \infty} || Tx_i - Tx_j || \leq || T_n - T || || x_i - x_j || \leq 2 || T_n - T ||
$$

Let $n \rightarrow \infty$.

Examples of compact operators.

1. $\mathcal{X} = C[0,1]$. $(T f)(s) = \int_0^1 K(s,t) f(t) dt$ for a nice continuous function $K$ of two variables.

2. Let $x_1, x_2, \ldots, x_n \in \mathcal{X}$, $\Lambda_1, \ldots, \Lambda_n \in \mathcal{X}^\ast$. $T x = \sum_{i=1}^{n} \Lambda_i(x)x_i$.

The adjoint. If $T : \mathcal{X} \rightarrow \mathcal{Y}$, $A^\ast : \mathcal{Y}^\ast \rightarrow \mathcal{X}^\ast$ is defined by

$$
< T^\ast y^\ast, x > = < y^\ast, Tx >
$$

$T$ bounded implies $T^\ast$ is bounded by the same bound.

$$
|| T || = \sup_{||x|| \leq 1} ||Tx|| = \sup_{||x|| \leq 1} |< Tx, y >| = \sup_{||x|| \leq 1} |< x, T^\ast y >| = \sup_{||x|| \leq 1, ||y^\ast|| \leq 1} ||T^\ast y|| = ||T^\ast||
$$

If $T$ is compact so is $T^\ast$. Let $K = T^\ast B_1$ the image of the unit ball. For any $\epsilon > 0$ we need to cover $K$ by a finite number balls of radius $\epsilon$. We can view $K \subset \mathcal{X}^\ast$ as functions on $\mathcal{X}$. If $x_1^\ast, x_2^\ast$ are two members of $K$, $||x_1^\ast - x_2^\ast|| = ||T^\ast y_1^\ast - T^\ast y_2^\ast||$ for some $y_1^\ast, y_2^\ast \in B_1(\mathcal{Y}^\ast)$.

$$
||T^\ast y_1^\ast - T^\ast y_2^\ast|| = \sup_{||x|| \leq 1} |< T^\ast (y_1^\ast - y_2^\ast), x >|
$$

$$
= \sup_{||x|| \leq 1} |(y_1^\ast - y_2^\ast), Tx >|
$$

$$
= \sup_{y \in TB_1(\mathcal{X})} |< y_1^\ast - y_2^\ast, y >|
$$

The linear functionals $< y^\ast, y >$ are continuous on the compact set $K_1 = TB_1(\mathcal{X}$ and satisfy a uniform estimate $|< y^\ast, y_1 - y_2 >| \leq ||y_1 - y_2||$. They are uniformly bounded. By Ascoli-Arzela theorem the space of functions is compact and can be covered by a finite number of balls.
Hilbert Spaces. A Hilbert space $H$ is a vector space with an inner product $\langle x, y \rangle$ that satisfies

1. $\langle x, y \rangle = \langle y, x \rangle$ is linear in $x$ for each $y$ and linear in $y$ for each $x$.
2. $\langle x, x \rangle > 0$ for $x \neq 0$.

It follows that

$$\langle (y + tx), (y + tx) \rangle = \langle y, y \rangle + 2t \langle x, y \rangle + t^2 \langle x, x \rangle \geq 0$$

and

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

and if we define $\|x\| = \sqrt{\langle x, x \rangle}$ then $|\langle x, y \rangle| \leq \|x\| \|y\|$ and $\|x\|$ is a norm on $H$.

3. The space $H$ is complete under the norm $\|x\|$.

Two vectors $x_1, x_2$ are orthogonal if $\langle x_1, x_2 \rangle = 0$. Denoted by $x_1 \perp x_2$.

A collection $\{x_\alpha\}$ is orthonormal if $x_\alpha \perp x_\beta$ for $\alpha \neq \beta$ and $\|x_\alpha\| = 1$ for all $\alpha$.

A complete orthonormal set is a maximal orthonormal collection $\{x_\alpha\}$ such that if $x \perp x_\alpha$ for $\alpha$ then $x = 0$.

We will assume that our Hilbert Space $H$ is separable. Since $\|x_\alpha - x_\beta\| = \sqrt{2}$ if $\alpha \neq \beta$ in an orthonormal set, any orthonormal set in a separable space has to be countable.

Given any set of $n$ mutually orthogonal vectors $x_1, x_2, \ldots, x_n \in H$, and a additional vector $y$ linearly independent of $x_1, x_2, \ldots, x_n$, there exists $x_{n+1} = c_{n+1}[y - \sum_{j=1}^{n} c_jx_j]$ such that $x_1, x_2, \ldots, x_n, x_{n+1}$ is a set of $n + 1$ orthonormal vectors and span the same subspace as $x_1, x_2, \ldots, x_n, y$. For $1 \leq j \leq n$, $\langle x_{n+1}, x_j \rangle = 0$ yields $\langle y, x_j \rangle = c_j$. We need to determine $c_{n+1}$. To this end

$$\langle x_{n+1}, x_{n+1} \rangle = c_{n+1}^2 \left[ \|y - \sum_{j=1}^{n} c_jx_j\|^2 \right] = 1$$

Finally need to check that $\|y\|^2 > \sum_{j=1}^{n} c_j^2$. Since $y$ is not in the span of $x_1, \ldots, x_n \|y - \sum_{j=1}^{n} c_jx_j\| > 0$. It follows that any separable Hilbert space has a countable orthonormal set that spans $H$, i.e an orthonormal basis. Start with a countable dense set and trim it to a linearly independent set that spans $H$ and then replace them inductively by an orthonormal set. This is known as the Gram-Schmidt process. You end with an orthonormal basis. Complete Orthonormal Set. $\{x_j\}$. If $y \perp x_j$ for all $j$ then $y = 0$.

$\{e_i\}$ is an orthonormal set of vectors. The following are equivalent

1. $\{e_i\}$ is maximal. That is if $x \perp e_i$ for all $i$ then $x = 0$
2. For any $y \in H$, $\|y\|^2 = \sum_i \langle y, e_i \rangle^2$
3. For any $y \in H$, $y = \sum_i \langle y, e_i \rangle e_i$
Proof. 3 ⇒ 2 ⇒ 1 is obvious. Need to prove 1 ⇒ 3

\[ \|y\|^2 \geq \sum_i <y, e_i>^2 \]

\[ <y - \sum_i <y, e_i> e_i, e_j> = 0 \]

for all \(j\). Therefore \(y - \sum_i <y, e_i> e_i = 0\) because of maximality.

The space \(l_2\). Sequences \(x = \{a_1, a_2, \ldots\}\) that are square summable, i.e \(\sum_{j=1}^{\infty} a_j^2 < \infty\).

\(<x, y> = \sum_{j=1}^{\infty} a_j b_j\)

Weak Convergence. \(<x_n, y> \rightarrow <x, y>\) for all \(y \in \mathcal{H}\)

If \(x_n\) converges weakly then \(\|x_n\|\) is bounded. An application of Baire Category Theorem.

\[ \mathcal{H} = \bigcup_k \{y : \sup_n |<x_n, y>| \leq k\} \]

For some \(k\), \(\{y : \sup_n |<x_n, y>| \leq k\}\) has interior. In other words for some \(x_0, k\) and \(\delta\)

\[ \sup_{\|y-x_0\|<\delta} \sup_n |<x_n, y>| \leq k \]

or

\[ \sup_{\|y\|<1} \sup_n |<x_n, y>| \leq \frac{2k}{\delta} \]

Unit Ball is weakly compact. \(<x, y>\) is jointly continuous in the strong or norm topology. \(<x_n, y_n> \rightarrow <x, y>\) if either \(x_n \rightarrow x\) strongly or \(y_n \rightarrow y\) strongly while the other can converge weakly. If both converge weakly it may not converge. In fact if \(x_n \rightarrow x\) weakly and \(\|x_n\| \rightarrow \|x\|\) then \(\|x_n - x\| \rightarrow 0\).

There is only one Hilbert Space of given dimension. Finite dimension \(d\). Countable infinite dimension. Any correspondence between complete orthonormal basis sets up an isomorphism. In particular \(\mathcal{H}^* = \mathcal{H}\). The adjoint \(T^*x\) is defined by \(<T^*x, y> = <x, Ty>\) for all \(y\). Self adjoint operators are those for which \(T^* = T\), or \(<Tx, y> = <x, T^*y> \quad \forall x, y\).

Eigen Values, Eigen functions etc. May not exist. Compact Self adjoint operators have a complete orthonormal set of eigen functions, with eigenvalues accumulating at 0.