**Riesz Representation Theorem.** Let $\Lambda(f)$ be a bounded linear functional on $C(X)$ the space of continuous functions on a compact metric space $X$. Then there is a signed measure $\mu$ on the Borel $\sigma$-field $\mathcal{B}$ of $X$, such that

$$\Lambda(f) = \int_X f(x) d\mu$$

This is done in several steps. $\Lambda$ is non-negative if for every $f \geq 0$, $\Lambda(f) \geq 0$.

First we need a result called partition of unity. We will deal only with functions that satisfy $0 \leq f \leq 1$. We always assume it is so.

**Lemma.** Let $X$ be compact metric space. Let $\{G_i\}$ be a finite collection open sets with $\bigcup_{i=1}^{n} G_i \supset C$ where $C$ is a closed set. Then there are nonnegative continuous functions $h_i$ with its support contained in $G_i$ such that $\sum_{i=1}^{n} h_i = 1$ on $C$.

**Proof.** For each $x \in C$ there is some open set $G_i$ that contains $x$, and therefore a ball $B(x, \delta(x))$ around $x$ of radius $\delta(x)$ whose closure $\bar{B}(x, \delta(x))$ is contained in $G_i$. Such balls provide a covering of $C$ and we extract a finite sub cover. Each ball is contained in some $G_i$ and we divide them in to $n$ groups depending on which $G_i$ it is contained in. If there are several possibilities choose any one. Let their unions be $W_i$ with closures $\overline{W_i} \subset G_i$. There are functions $g_i$ that are 1 on $W_i$ with support contained in $G_i$. We define

$$h_1 = g_1, h_2 = g_2(1 - g_1), \ldots, h_n = g_n(1 - g_1) \cdots (1 - g_{n-1})$$

Then

$$h_1 + h_2 + \cdots + h_n = 1 - (1 - g_1) \cdots (1 - g_n)$$

Since some $g_i = 1$ at every point of $C$ we are done.

1. Any bounded $\Lambda$ can be written as $\Lambda^+ - \Lambda^-$ where $\lambda^\pm$ are both non-negative.

**Proof.** For $f \geq 0$, define

$$\Lambda^+(f) = \sup_{0 \leq g \leq f} \Lambda(g)$$

$$\Lambda^+(f_1 + f_2) = \Lambda^+(f_1) + \Lambda^+(f_2)$$

For $c > 0$

$$\Lambda^+(cf) = c\Lambda^+(f)$$

For arbitrary $f$ we write $f = (f + C) - C$ and $\Lambda^+(f) = \Lambda^+(f + C) - \Lambda^+(C)$. It is well defined does not depend on $C$.

One defines $\Lambda^-(f) = \Lambda^+(f) - \Lambda(f)$ so that for $f \in C(X)$, $\Lambda(f) = \Lambda^+(f) - \Lambda^-(f)$. It is easy to verify that for $f \geq 0$, $\Lambda^-(f) \geq 0$ because $\Lambda^+(f) \geq \Lambda(f)$. 1
\[\|\Lambda^+\| + \|\Lambda^-\| = \Lambda^+(1) + \Lambda^-(1)\]
\[= \Lambda^+(1) + \Lambda^+(1) - \Lambda(1)\]
\[= \sup_{0 \leq f \leq 1} \Lambda(f) + \sup_{0 \leq g \leq 1} \Lambda(g) - \Lambda(1)\]
\[= \sup_{0 \leq f \leq 1} \Lambda(f) + \sup_{0 \leq g \leq 1} \Lambda(g) - \Lambda(1)\]
\[= \sup_{0 \leq f \leq 1} \Lambda(f) + \sup_{0 \leq g \leq 1} \Lambda(-g)\]
\[= \sup_{0 \leq f \leq 1} \Lambda(f - g) = \sup_{0 \leq |f| \leq 1} \Lambda(f) = \|\Lambda\|\]

The problem now is reduced to proving that a non-negative linear functional which is bounded by \(\Lambda(1)\) has the representation in terms of a non-negative measure \(\mu\).

\[\Lambda(f) = \int_X f(x) \, d\mu\]

2. For any open set \(G\) we define

\[\mu(G) = \sup_{0 \leq f \leq 1 \atop \text{support } f \subset G} \Lambda(f)\]

Support \(f\) is \(\{x : f(x) \neq 0\}\).

**Remark.** We could take the sup over the larger class of \(f\) with \(f = 0\) on \(G^c\). Then \(\{x : f(x) \leq \epsilon\}\) will be a closed set in \(G\). And \(g = (f - \epsilon)^+\) with \(\Lambda(g) \geq \Lambda(f) - \epsilon\) can replace \(f\). The supremum will be the same.

3. For any set \(E\) we define

\[\mu(E) = \inf_{G \supset E \atop G \text{ open}} \mu(G)\]

4 We say \(E \in \Sigma\) if

\[\mu(E) = \sup_{C \subset E \atop C \text{ closed}} \mu(C)\]

5. If \(\{E_i\}\) is any countable collection of subsets of \(X\)

\[\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)\]

**Proof.** Let us first show that if \(G_1, G_2\) are open

\[\mu(G_1 \cup G_2) \leq \mu(G_1) + \mu(G_2)\]
Given $\epsilon > 0$, there is a function $g_\epsilon(x)$, $0 \leq g_\epsilon \leq 1$, with support $C_\epsilon$ contained in $G_1 \cup G_2$ with $\Lambda(g_\epsilon) \geq \mu(G) - \epsilon$. There are two non-negative functions $h_1, h_2$ with their supports contained in $G_1$ and $G_2$ with $h_1 + h_2 = 1$ on $C_\epsilon$. $g_\epsilon = g_\epsilon h_1 + g_\epsilon h_2$.

$$\mu(G_1) + \mu(G_2) \geq \Lambda(g_\epsilon h_1) + \Lambda(g_\epsilon h_2) = \Lambda(g_\epsilon) \geq \mu(G_1 \cup G_2) - \epsilon$$

We can assume that $\sum_i \mu(E_i) < \infty$. Pick open sets $V_i \supset E_i$ such that $\mu(V_i) \leq \mu(E_i) + \epsilon 2^{-i}$. Let $V = \bigcup_i V_i$. Let $f$ be such that the support $D$ of $f$ is contained in $V$ and $\Lambda(f) \geq \mu(V) - \epsilon$.

Since $D$ is compact and contained in $V$ it is contained in $\bigcup_{i=1}^n V_i$ for some finite $n$.

$$\Lambda(f) \leq \mu(\bigcup_{i=1}^n V_i) \leq \sum_{i=1}^n \mu(E_i) + \epsilon \leq \sum_{i=1}^\infty \mu(E_i) + \epsilon$$

Since this is true for every $f$ with support contained in $V$

$$\Lambda(E) \leq \Lambda(V) \leq \sum_{i=1}^\infty \mu(E_i) + \epsilon$$

$\epsilon$ is arbitrary.

6. If $C$ is a closed set then

$$\mu(C) = \inf \{\Lambda(f) : 0 \leq f \leq 1, f = 1 \text{ on } C\}$$

**Proof.** Since

$$\mu(C) = \inf \{\mu(V) : V \text{ open ; } V \supset C\}$$

We need to show two things.

Given any $f$, such that $f = 1$ on $C$, for any $\epsilon > 0$, $\{x : f(x) > 1 - \epsilon\}$ is an open set $V_\epsilon \supset C$. If $g$ is any function supported in $V_\epsilon$, $(1 - \epsilon)g \leq f$ or $\Lambda(g) \leq (1 - \epsilon)^{-1}\Lambda(f)$. Since $\mu(V_\epsilon) = \{\sup \Lambda(g) : \text{ support } g \subset V_\epsilon\}$ it follows that $\mu(V_\epsilon) \leq (1 - \epsilon)^{-1}\Lambda(f)$.

In the reverse direction given $V \supset C$ by Urysohn’s lemma there is an $f$ that is 1 on $C$ with support inside $V$. Then $\Lambda(f) \leq \mu(V)$.

7. If $G$ is open

$$\mu(G) = \sup_{C \subset G} \mu(C)$$

**Proof.** Since

$$\mu(G) = \sup \{\Lambda(g) : 0 \leq g \leq 1; \text{ support } g \subset G\}$$

We need to show two things.

$G$ is an open set and $0 \leq g \leq 1$ is a function supported on a closed subset $C$ of $G$. If $f = 1$ on $C$, then $f \geq g$ and $\Lambda(f) \geq \Lambda(g)$. If $W \supset C$ is any open set there is an $f$ that is 1 on $C$ and supported in $W$. Makes $\mu(W) \geq \Lambda(g)$. True for every $W \supset C$. Follows that $\mu(C) \geq \Lambda(g)$.
Conversely if \( C \subset G \) is any closed subset of \( G \), there is a function \( g \), \( 0 \leq g \leq 1 \) with support contained in \( G \) and

\[
\Lambda(g) \geq \mu(G) - \epsilon \geq \mu(C) - \epsilon
\]

8. If \( \{E_i\} \) are in \( \Sigma \) and pairwise disjoint \( E = \bigcup_{i=1}^{\infty} E_i \in \Sigma \) and

\[
\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)
\]

**Proof.** Let \( C_1 \) and \( C_2 \) be closed sets that are disjoint. There is a function \( f \), \( 0 \leq f \leq 1 \), \( f = 1 \) on \( C_1 \) and 0 on \( C_2 \). Let \( g \) equal 1 on \( C_1 \cup C_2 \) with \( \Lambda(g) \leq \mu(C_1 \cup C_2) + \epsilon \).

\[
\Lambda(gf) \geq \mu(C_1) \text{ and } \Lambda(g(1-f)) \geq \mu(C_2).
\]

Adding \( \mu(C_1 \cup C_2) + \epsilon \geq \Lambda(g) \geq \mu(C_1) + \mu(C_2) \) Letting \( \epsilon \to 0 \), \( \mu(C_1 \cup C_2) \geq \Lambda(g) \geq \mu(C_1) + \mu(C_2) \). We already have the other half.

Since \( E_i \in \Sigma \) there are closed sets \( D_i \subset E_i \) with \( \mu(D_i) \geq \mu(E_i) - \epsilon 2^{-i} \). \( \{D_i\} \) are pairwise disjoint as well.

\[
\mu(E) \geq \mu(\bigcup_{i=1}^{n} E_i) \geq \mu(\bigcup_{i=1}^{n} D_i) = \sum_{i=1}^{n} \mu(D_i) \geq \sum_{i=1}^{n} [\mu(E_i) - \epsilon \sum 2^{-i}]
\]

with \( n \to \infty \) and \( \epsilon \to 0 \)

\[
\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)
\]

We have the other half. Easy to check that \( E \in \mathcal{M} \). \( G_i \supset E_i \supset D_i \), \( \mu(G_i) - \mu(C_i) \leq \epsilon 2^{-i} \). Then \( \bigcup_{i=1}^{\infty} G_i \supset \bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} D_i \).

9. For any \( E \in \Sigma \) and any \( \epsilon > 0 \) there is an open set \( G \) and a closed set \( C \) such that \( C \subset E \subset G \) and \( \mu(G - C) \leq \epsilon \).

**Proof.** From our definitions we can find \( C \) and \( G \) such that \( C \subset E \) and \( E \subset G \) and

\[
\mu(C) \geq \mu(E) - \frac{\epsilon}{2}; \quad \mu(G) - \mu(E) \leq -\frac{\epsilon}{2}
\]

\( G = C \cup (G - C) \) is a disjoint union and both are in \( \Sigma \). \( \mu(G) = \mu(C) + \mu(G - C) \). Therefore \( \mu(G - C) \leq \epsilon \).

10. \( \Sigma \) is a Field.

**Proof.** If \( E_1, E_2 \in \Sigma \), for any \( \epsilon > 0 \) can find \( C_1, C_2, G_1, G_2 \) such that \( C_i \subset E_i \subset G_i \) and \( \mu(G_i - C_i) \leq \frac{\epsilon}{2} \). \( \left( (G_1 \cup G_2) - (C_1 \cup C_2) \right) \subset \left( (G_1 - C_1) \cup (G_2 - C_2) \right) \). \( \mu((G_1 - C_1) \cup (G_2 - C_2)) \leq \epsilon \). \( (E_1 \cup E_2) \in \Sigma \). Similarly intersection and complementation.

11. \( \Sigma \) is sigma field and \( \mu \) is a measure on \( \Sigma \).

**Proof.** Done.
12. $\int f \, d\mu = \Lambda(f)$

**Proof.** It is enough to prove $\Lambda(f) \leq \int f \, d\mu$. We can add constants to both sides $\Lambda(1) = \mu(X)$. Can assume $f \geq 0$. Divide by a constant $0 \leq f \leq 1$.

Let $\epsilon > 0$ be given. Let $\{0 = y_0 < y_1 < \cdots < y_n = 1\}$ be the interval $[0, 1]$ divided into $n$ equal parts such that $\frac{1}{n} < \epsilon$. Let $E_i = \{x : y_{i-1} < f(x) \leq y_i\}$. We can include $f^{-1}(0)$ in $E_1$. $E_i$ are disjoint sets, $X = \cup_i E_i$. There are open sets $G_i \supset E_i$ with $\mu(G_i) < \mu(E_i) + \epsilon \frac{1}{n}$ and $f(x) \leq y_i + \epsilon$. Since $\{G_i\}$ is a covering of $X$, there is a partition of unity $\{h_i\}$ with $\sum_i h_i = 1$, and $h_i$ supported inside $G_i$. We have $f = \sum_i h_i f$. Note that $\Lambda(h_i) \leq \mu(G_i) \leq \mu(E_i) + \epsilon \frac{1}{n}$.

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \epsilon) \Lambda(h_i) \leq \sum_{i=1}^{n} (y_i - \epsilon + 2\epsilon) \mu(E_i) + \epsilon \frac{1}{n} + 2\epsilon$$

$$\leq \sum_{i=1}^{n} (y_i - \epsilon) \mu(E_i) + 2\epsilon + \epsilon(1 + \epsilon) \leq \int f \, d\mu + 3\epsilon + \epsilon^2$$

**Dual of $L_p$ spaces.** Let $(\Omega, \Sigma, \mu)$ be a measure space where $\mu$ is a finite measure on the $\sigma$-field $\Sigma$ of subsets of $\Omega$. $X$ be the Banach space $L_p(\Omega, \Sigma, \mu)$ of $\Sigma$ measurable functions that satisfy $\int_{\Omega} |f(\omega)|^p \, d\mu < \infty$ with the norm

$$\|f\|_p = \left[ \int_{\Omega} |f(\omega)|^p \, d\mu \right]^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Let $\Lambda(f)$ be a bounded linear functional on $L_p(\Omega, \Sigma, \mu)$. If $1 < p < \infty$

$$\Lambda(f) = \int fg \, d\mu$$

for some $g \in L_q(\Omega, \Sigma, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. $\|\Lambda\| = \|g\|_q$. If $p = 1$, $q = \infty$ and it is still true that

$$\Lambda(f) = \int fg \, d\mu$$

but $g \in L_\infty(\Omega, \Sigma, \mu)$. $L_\infty$ consists of functions $g$ that are essentially bounded, i.e. there is a bound $M$ such that $\mu[\omega : |g(\omega)| > M] = 0$. $\|g\|_\infty$ is the smallest $M$ that works. $\|\Lambda\| = \|g\|_\infty$. Since

$$|\int fg \, d\mu| \leq \|f\|_p \|g\|_q$$

for conjugate pairs $p, q$ the functions $g$ in $L_q$ do define bounded linear functionals with the correct bound. We concentrate now on the converse. Since $\mu$ is a finite measure, $1_A(\omega) \in L_p$. Define

$$\lambda(A) = \Lambda(1_A(\omega))$$

$$\|\lambda(A)\| \leq C \|1_A(\omega)\|_p$$
\[ \sup_{A \in \Sigma} \lambda(A) \leq C \sup_{A \in \Sigma} \|1_A(\omega)\|_p = C[\mu(\Omega)]^{\frac{1}{p}} \]

To prove \( \lambda \) is a countably additive signed measure, we need to check that for a countable collection of pairwise disjoint sets \( A_i \), with \( \bigcup_{i=1}^\infty A_i = A \), we have

\[ \left\| \left( \sum_{i=1}^n 1_{A_i}(\omega) \right) - 1_A(\omega) \right\|_p \to 0. \]

The difference is the indicator of the set \( \bigcup_{i=n+1}^\infty A_i \) whose measure tends to 0 and so does its \( L_p \) norm for \( 1 \leq p < \infty \). \( \lambda(A) \) is a signed measure. \( \lambda \ll \mu \). There is a Radon-Nikodym derivative.

\[ \Lambda(1_A) = \lambda(A) = \int_A g d\mu \]

with \( g \in L_1 \). \( \Lambda(f) = \int f g d\mu \) for simple functions and then for bounded measurable functions. Take \( f = (\operatorname{sign} g)|g|^{q-1}1_{|g| \leq M} \). Then \( f \) is bounded and \( pq = p + q \)

\[ \int |f|^p d\mu = \int_{|g| \leq M} |g|^{pq-p} d\mu = \int_{|g| \leq M} |g|^q d\mu \]

\[ \Lambda(f) = \int_{|g| \leq M} |g|^q d\mu \leq C \left[ \int_{|g| \leq M} |g|^q d\mu \right]^{\frac{1}{p}} \]

\[ \left[ \int_{|g| \leq M} |g|^q d\mu \right]^{\frac{1}{q}} \leq C \]

Let \( M \to \infty \). \( g \in L_q \) and \( \|g\|_q \leq C \)

If \( p = 1 \), \( |\lambda(A)| \leq C \mu(A) \). \( g = \frac{dA}{d\mu} \) \( |g| \leq C \) a.e. or \( \|g\|_\infty \leq C \).

**\( \ell_p \) spaces.** The space of sequences \( \xi = \{a_n\} : n \geq 1 \).

\[ \|\xi\|_p = \left[ \sum_{i=1}^\infty |a_n|^p \right]^{\frac{1}{p}} \]

The Dual of \( \ell_p \) is \( \ell_q \). \( pq = p + q \). \( \|\xi\|_\infty = \sup_n |a_n| \).