**Banach Spaces.** $\mathcal{X}$ is called a Banach Space if $\mathcal{X}$ is a vector space. There is a function $\|x\|$ called norm defined on $\mathcal{X}$ with the following properties.

1. $\|x\| \geq 0$, $\|x\| = 0 \iff x = 0$. $\|cx\| = |c|\|x\|$, $\|x + y\| \leq \|x\| + \|y\|$

2. This makes $d(x, y) = \|x - y\|$ a metric on $\mathcal{X}$ and $(\mathcal{X}, d)$ is a complete metric space.

**Examples.**

1. $\mathcal{X} = \mathbb{R}^d$, $\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}$.

2. $\mathcal{X} = C(M)$ the space of bounded continuous functions on $(M, d)$. $f \in \mathcal{B}$ is a bounded continuous function on $M$ and $\|f\| = \sup_{x \in M} |f(x)|$.

3. For $1 \leq p < \infty$, $\mathcal{B} = L^p(\Omega, \Sigma, \mu)$, the space of measurable functions $f$ such that $|f|^p$ is integrable.

$$\|f\|_p = \left[ \int_{\Omega} |f(\omega)|^p d\mu \right]^\frac{1}{p}$$

A linear map $T$ from one Banach space $\mathcal{X}$ to $\mathcal{Y}$ is bounded if for some $C < \infty$, $\|Tx\| \leq C\|x\|$ for all $x \in \mathcal{X}$. The smallest $C$ that works is denoted by $\|T\|$.

A linear map is continuous if it is continuous at 0. And it is continuous at 0 if and only if it is bounded. $\|Tx_n - Tx\| = \|T(x_n - x)\| \leq C\|x_n - x\|$. Shows that boundedness implies continuity and continuity at 0 implies continuity everywhere. Finally if it is not bounded we can find $x_n \in \mathcal{X}$ such that $\|\frac{x_n}{\|Tx_n\|}\| = \frac{\|x_n\|}{\|Tx_n\|} \to 0$. But $\|T\frac{x_n}{\|Tx_n\|}\| = 1$

If $T$ is bounded, one to one and maps $\mathcal{X}$ onto $\mathcal{Y}$ its inverse is bounded. There is a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ or $\|T^{-1}x\| \leq c^{-1}\|x\|$. What we need to prove is that he image of the unit ball $\|x\| \leq 1$ under $T$ contains a ball $\|y\| \leq c$ for some $c$. Then $T^{-1}\{\|y\| \leq c\}$ will be contained in the unit ball of $\mathcal{X}$. Makes $\|T^{-1}\| \leq c^{-1}$.

Denoting by $B(a, r) = \{x : \|x - a\| \leq r\}$ we have

$$\bigcup_{n=1}^{\infty} TB(0, n) = T\mathcal{X} = \mathcal{Y}$$

Some $\overline{TB(0, n)}$ must have a ball $B(p, r)$ by Baire Category Theorem. Then

$$\overline{TB(0, 2n)} \supset \overline{TB(0, n)} - \overline{TB(0, n)} \supset B(p, r) - B(p, r) \supset B(0, 2r)$$

Then by homogeneity for some $\eta > 0$

$$\overline{TB(0, 1)} \supset B(0, \eta)$$

Let $y \in B(0, \eta)$ be given. There is $x_1 \in B(0, 1)$ such that

$$\|Tx_1 - y\| \leq \frac{\eta}{2}$$
There is now an \( x_2 \in B(0, \frac{1}{2}) \) such that
\[
\|Tx_1 - Tx_2 - y\| \leq \frac{\eta}{2^2}
\]
Inductively there is \( x_n \in B(0, 2^{-n}) \) such that
\[
\|\sum_{i=1}^{n} Tx_i - y\| \leq \frac{\eta}{2^n}
\]
Clearly \( \sum_i x_i = x \) exists \( \|x\| \leq 2 \) and \( Tx = y \). Image of the unit ball contains a ball around the origin. Makes inverse bounded.

If a Banach space \( \mathcal{X} \) is complete under each of two norms \( ||x|| \) and \( |||x||| \), and if \( ||x|| \leq C|||x||| \) then \( |||x||| \leq C'|||x||| \) with another constant \( C' \). Let the Banach space with stronger norm be \( \mathcal{X} \) and the one weaker norm \( \mathcal{Y} \), the identity map \( T \) from \( \mathcal{X} \to \mathcal{Y} \) is bounded, one to one and onto. The inverse is therefore bounded.

**Bounded Linear Functionals.** Maps \( \Lambda \) from a Banach Space \( \mathcal{X} \to \mathbb{R} \) such that \( |\lambda(x)| \leq C||x|| \) The smallest \( C \) is called \( \|\lambda\| \). Makes such linear functionals into a Banach space with norm
\[
\|\lambda\| = \sup_{||x|| \leq 1} |\lambda(x)|
\]

**Hahn-Banach Theorem.** Given a subspace \( \mathcal{Y} \subset \mathcal{X} \) and a bounded linear functional \( \Lambda \) on \( \mathcal{Y} \) with bound \( |\lambda(x)| \leq c||x|| \) for all \( x \in \mathcal{Y} \), it can be extended as a bounded linear functional on \( \mathcal{X} \) satisfying \( |\lambda(x)| \leq c||x|| \) for all \( x \in \mathcal{X} \) with the same bound \( c \).

**Proof.** Let us take \( x_0 \notin \mathcal{Y} \) and consider \( x + ax_0 \) with \( x \in \mathcal{Y} \) and a scalar \( a \). We define \( \Lambda(x + ax_0) = \Lambda(x) + a\theta \) for some \( \theta \). We want to pick it so that for all \( x \in \mathcal{Y} \) and \( a \in \mathbb{R} \)
\[
|\Lambda(x) + a\theta| \leq c||x + ax_0||
\]
This means
\[
-c||x + ax_0|| \leq \Lambda(x) + a\theta \leq c||x + ax_0||
\]
Take \( a > 0 \).
\[
\frac{-\Lambda(x) - c||x + ax_0||}{a} \leq \theta \leq \frac{-\Lambda(x) + c||x + ax_0||}{a}
\]
Needs for all \( x, y \in \mathcal{Y} \)
\[
\frac{-\Lambda(y) - c||y + ax_0||}{a} \leq \theta \leq \frac{-\Lambda(x) + c||x + ax_0||}{a}
\]
From \( a < 0 \) we get
\[
\frac{\Lambda(y) - c||y - ax_0||}{a} \leq \theta \leq \frac{\Lambda(x) + c||x - ax_0||}{a}
\]
They are both the same. Need for all $x, y \in \mathcal{Y}$

$$\Lambda(y) - c\|y - ax_0\| \leq \Lambda(x) + c\|x - ax_0\|$$

But

$$|\Lambda(x) - \Lambda(y)| \leq c\|x - y\| = c\|(x - x_0) - (y - y_0)\| \leq c\|x - x_0\| + c\|y - x_0\|$$

We can extend by one step. Induction. Extends to closure.

$(M, d)$ is a compact metric space. $\mathcal{X} = C(M)$ is the space of continuous functions on $(M, d)$ which are bounded because $M$ is compact. $\mathcal{X}$ is a Banach space with the norm $\|f\| = \sup_{x \in M} |f(x)|$.

**Stone-Weierstrass Theorem.**

Let $\mathcal{A} \subset \mathcal{X}$ be a sub algebra of continuous functions that contains constants and given any two points $x, y$ in $M$, there is a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then $\mathcal{A}$ is dense in $\mathcal{X}$.

**Examples.**

1. $X = [0, 1]$. Polynomials are dense in $C[0, 1]$.

2. $\{\cos nx, \sin nx\}$ are dense $C(S)$. Periodic continuous functions on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

**Proof.** $\mathcal{A}$ is an algebra in $C(X)$ that is closed. Then for any $f \in \mathcal{A}$, $|f| \in \mathcal{A}$. We can assume $|f| \leq 1$. Then $0 \leq 1 - f^2 \leq 1$. The power series expansion for $(1 - x)^{1/2}$ converges uniformly on $0 \leq x \leq 1$. $(1 - (1 - f^2))^{1/2} = |f|$ is a convergent power series in $(1 - f^2)$ and is therefore a uniform limit of polynomials in $f^2$ or $f$. It is in $\mathcal{A}$. It now follows that $f \wedge g$ and $f \vee g$ are also in $\mathcal{A}$. To see it we note

$$f \wedge g = \frac{f + g - |f - g|}{2}, \quad f \vee g = \frac{f + g + |f - g|}{2}$$

The problem reduces to the following. Given a continuous function $g \in C(X)$ and an $\epsilon > 0$, need to produce a function $f$ from $\mathcal{A}$ satisfying

$$g(x) - \epsilon \leq f(x) \leq g(x) + \epsilon$$

for all $x$. Let $a, b$ be two different points in $X$. There is function function $f$ that separates them, i.e. $f(a) \neq f(b)$. By taking linear combination with constants, i.e. a function of the form $\alpha f + \beta$ we can match $f(a) = g(a)$ and $f(b) = g(b)$. Let us call this function $f_{ab}(x)$. $f_{ab} \in \mathcal{A}$ and $f_{ab}(a) = g(a), f_{ab}(b) = g(b)$. By continuity for $b \neq a$, there is an open set $N_{ab}$ containing $a, b$, and $g(x) - \epsilon \leq f_{ab}(x) \leq g(x) + \epsilon$ on $N_{ab}$. For fixed $a, \cup_{b \neq a} N_{ab} = X$ and there is a finite sub cover with $b \in F$. Let $f^*_a(x) = \wedge_{b \in F} f_{ab}(x)$. Then $f^*_a(x) \leq g(x) + \epsilon$ for all $x \in X$ and

$$g(x) - \epsilon \leq f^*_a(x)$$
on \( N_a = \cap_{b \in F} N_{ab} \) which is open and contains \( a \). Since \( \{N_a\} \) is an open covering and there is a finite sub cover \( G \)

\[
f^{**} = \vee_{a \in G} f^*_a
\]

will work to give

\[
g(x) - \epsilon \leq f^{**}(x) \leq g(x) + \epsilon
\]

**Compact subsets of \( C(X) \)**

**Ascoli-Arzela Theorem.** A closed subset \( K \subset C(X) \) of continuous functions on a compact metric space \( X \) is compact if and only if

\[
\sup \sup |f(x)| < \infty
\]

and

\[
\lim_{\delta \to 0} \sup \sup_{d(x,y) \leq \delta} |f(x) - f(y)| = 0
\]

**Remark.** The function

\[
\omega_f(\delta) = \sup_{d(x,y) \leq \delta} |f(x) - f(y)|
\]

is called the modulus of continuity of \( f \) and tends to 0 as \( \delta \to 0 \). It is uniform over any compact set of continuous functions. In fact there is **Dini’s Theorem**.

If \( X \) is compact and \( f_n(x) \) are continuous functions and \( f_n(x) \downarrow 0 \) then \( f_n \to 0 \) uniformly on \( X \).

**Proof.** Given \( \epsilon > 0 \) and \( x \in X \) there is \( n_0(x) \) such that \( f_{n_0(x)}(x) < \frac{\epsilon}{2} \). By continuity \( f_{n_0(x)}(y) < \epsilon \) in a neighborhood \( N_x \). \( \{N_x\} \) is a covering. take a finite subcover \( x \in F \). By monotonicity \( f_n(x) \leq \epsilon \) for all \( x \) and \( n \geq \sup_{x \in F} n_0(x) \).

The necessity of Ascoli-Arzela theorem is obvious. \( ||f|| = \sup_x |f(x)| \) is continuous and has to be bounded on compact sets. The modulus continuity \( \omega_f(\delta) \), tends to 0 monotonically and has to be uniform by Dini’s theorem.

**Sufficiency.** Let \( D \) be a countable dense subset of \( X \). Given a bounded sequence from \( K \) we can choose by diagonalization a subsequence \( f_n(x) \) that converges at every point of \( D \) to a limit \( f(x) \) defined for \( x \in D \). In particular for \( x \in D \), as \( n, m \to \infty \)

\[
|f_n(x) - f_m(x)| \to 0
\]

Let \( \epsilon > 0 \) be given. \( |f_n(x) - f_n(y)| < \epsilon \) if \( d(x,y) < \delta \). Since \( X \) is compact there is a finite set \( F \) from \( D \) such that for any \( x \in X \), there is a \( y \in F \) with \( d(x,y) < \delta \) making \( |f_n(x) - f_n(y)| < \epsilon \) for all \( n \). Then

\[
|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(x) - f_m(y)|
\]

\[
\leq \sup_{y \in F} |f_n(y) - f_m(y)| + 2\epsilon
\]

Makes \( f_n \) Cauchy in \( C(X) \).