Banach Spaces. $\mathcal{X}$ is called a Banach Space if $\mathcal{X}$ is a vector space. There is a function $\|x\|$ called norm defined on $\mathcal{X}$ with the following properties.

1. $\|x\| \ge 0$, $\|x\| = 0 \Leftrightarrow x = 0$. $\|cx\| = |c|\|x\|$. $\|x + y\| \le \|x\| + \|y\|$

2. This makes $d(x, y) = \|x - y\|$ a metric on $\mathcal{X}$ and $(\mathcal{X}, d)$ is a complete metric space.

Examples.

1. $\mathcal{X} = R^d$. $\|x\| = \sqrt{\sum_{i=1}^d x_i^2}$.

2. $\mathcal{X} = C(M)$ the space of bounded continuous functions on (M, d). $f \in \mathcal{B}$ is a bounded continuous function on M and $\|f\| = \sup_{x\in M} |f(x)|$.

3. For $1 \le p < \infty$, $\mathcal{B} = L_p(\Omega, \Sigma, \mu)$, the space of measurable functions f such that $|f|^p$ is integrable.

$$\|f\|_p = \left[\int_\Omega |f(\omega)|^p d\mu\right]^{\frac{1}{p}}$$

A linear map T from one Banach space $\mathcal{X}$ to $\mathcal{Y}$ is bounded if for some $C < \infty$, $\|Tx\| \le C\|x\|$ for all $x \in \mathcal{X}$. The smallest C that works is denoted by $\|T\|$

A linear map is continuous if it is continuous at 0. And it is continuous at 0 if and only if it is bounded. $\|Tx_n - Tx\| = \|T(x_n - x)\| \le C\|x_n - x\|$. Shows that boundedness implies continuity and continuity at 0 implies continuity everywhere. Finally if it is not bounded we can find $x_n \in \mathcal{X}$ such that $\|\frac{x_n}{\|Tx_n\|}\| = \frac{\|x_n\|}{\|Tx_n\|} \to 0$. But $\|T\frac{x_n}{\|Tx_n\|}\| = 1$

If T is bounded, one to one and maps $\mathcal{X}$ onto $\mathcal{Y}$ its inverse is bounded. There is a constant $c > 0$ such that $\|Tx\| \ge c\|x\|$ or $\|T^{-1}x\| \le c^{-1}\|x\|$. What we need to prove is that he image of the unit ball $\|x\| \le 1$ under T contains a ball $\|y\| \le c$ for some c. Then $T^{-1}\{\|y\| \le c\}$ will be contained in the unit ball of $\mathcal{X}$. Makes $\|T^{-1}\| \le c^{-1}$.

Denoting by $B(a, r) = \{x : \|x - a\| \le r\}$ we have

$$\cup_{n=1}^\infty TB(0, n) = T\mathcal{X} = \mathcal{Y}$$

Some $\overline{TB(0, n)}$ must have a ball $B(p, r)$ by Baire Category Theorem. Then

$$\overline{TB(0, 2n)} \supset \overline{TB(0, n)} - \overline{TB(0, n)} \supset B(p, r) - B(p, r) \supset B(0, 2r)$$

Then by homogeneity for some $\eta > 0$

$$\overline{TB(0, 1)} \supset B(0, \eta)$$

Let $y \in B(0, \eta)$ be given. There is $x_1 \in B(0, 1)$ such that

$$\|Tx_1 - y\| \le \frac{\eta}{2}$$

There is now an $x_2 \in B(0, \frac{1}{2})$ such that

$$\|Tx_1 - Tx_2 - y\| \leq \frac{\eta}{2^2}$$

Inductively there is $x_n \in B(0, 2^{-n})$ such that

$$\| \sum_{i=1}^{n} Tx_i - y\| \leq \frac{\eta}{2^n}$$

Clearly $\sum_i x_i = x$ exists $\|x\| \leq 2$ and $Tx = y$. Image of the unit ball contains a ball around the origin. Makes inverse bounded.

If a Banach space $\mathcal{X}$ is complete under each of two norms $||x||$ and $|||x|||$, and if $||x|| \leq C|||x|||$ then $|||x||| \leq C'||x||$ with another constant C'. Let the Banach space with stronger norm be $\mathcal{X}$ and the one weaker norm $\mathcal{Y}$, the identity map T from $\mathcal{X} \to \mathcal{Y}$ is bounded, one to one and onto. The inverse is therefore bounded.

Bounded Linear Functionals. Maps Λ from a Banach Space $\mathcal{X} \to R$ such that $|\lambda(x)| \leq C\|x\|$ The smallest C is called $\|\lambda\|$. Makes such linear functionals into a Banach space with norm

$$\|\lambda\| = \sup_{\|x\| \leq 1} |\lambda(x)|$$

Hahn-Banach Theorem. Given a subspace $\mathcal{Y} \subset \mathcal{X}$ and a bounded linear functional Λ on $\mathcal{Y}$ with bound $|\lambda(x)| \leq c\|x\|$ for all $x \in \mathcal{Y}$, it can be extended as a bounded linear functional on $\mathcal{X}$ satisfying $|\lambda(x)| \leq c\|x\|$ for all $x \in \mathcal{X}$ with the same bound c.

Proof. Let us take $x_0 \notin \mathcal{Y}$ and consider $x + ax_0$ with $x \in \mathcal{Y}$ and a scalar a. We define $\Lambda(x + ax_0) = \Lambda(x) + a\theta$ for some θ. We want to pick it so that for all $x \in \mathcal{Y}$ and $a \in R$

$$|\Lambda(x) + a\theta| \leq c\|x + ax_0\|$$

This means

$$-c\|x + ax_0\| \leq \Lambda(x) + a\theta \leq c\|x + ax_0\|$$

Take $a > 0$.

$$\frac{-\Lambda(x) - c\|x + ax_0\|}{a} \leq \theta \leq \frac{-\Lambda(x) + c\|x + ax_0\|}{a}$$

Needs for all $x, y \in \mathcal{Y}$

$$\frac{-\Lambda(y) - c\|y + ax_0\|}{a} \leq \theta \leq \frac{-\Lambda(x) + c\|x + ax_0\|}{a}$$

From $a < 0$ we get

$$\frac{\Lambda(y) - c\|y - ax_0\|}{a} \leq \theta \leq \frac{\Lambda(x) + c\|x - ax_0\|}{a}$$

They are both the same. Need for all $x, y \in \mathcal{Y}$

$$\Lambda(y) - c\|y - ax_0\| \leq \Lambda(x) + c\|x - ax_0\|$$

But

$$|\Lambda(x) - \Lambda(y)| \leq c\|x - y\| = c\|(x - x_0) - (y - y_0)\| \leq c\|x - x_0\| + c\|y - x_0\|$$

We can extend by one step. Induction. Extends to closure.

(M, d) is a compact metric space. $\mathcal{X} = C(M)$ is the space of continuous functions on (M, d) which are bounded because M is compact. $\mathcal{X}$ is a Banach space with the norm $\|f\| = \sup_{x \in M} |f(x)|$.

Stone-Weierstrass Theorem.

Let $\mathcal{A} \subset \mathcal{X}$ be a sub algebra of continuous functions that contains constants and given any two points x, y in M, there is a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then $\mathcal{A}$ is dense in $\mathcal{X}$.

Examples.

1. $X = [0, 1]$. Polynomials are dense in $C[0, 1]$.

2. $\{\cos nx, \sin nx\}$ are dense $C(S)$. Periodic continuous functions on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

Proof. $\mathcal{A}$ is an algebra in $C(X)$ that is closed. Then for any $f \in \mathcal{A}$, $|f| \in \mathcal{A}$. We can assume $|f| \leq 1$. Then $0 \leq 1 - f^2 \leq 1$. The power series expansion for $(1 - x)^{\frac{1}{2}}$ converges uniformly on $0 \leq x \leq 1$. $(1 - (1 - f^2))^{\frac{1}{2}} = |f|$ is a convergent power series in $(1 - f^2)$ and is therefore a uniform limit of polynomials in f^2 or f. It is in $\mathcal{A}$. It now follows that $f \wedge g$ and $f \vee g$ are also in $\mathcal{A}$. To see it we note

$$f \wedge g = \frac{f + g - |f - g|}{2}, \quad f \vee g = \frac{f + g + |f - g|}{2}$$

The problem reduces to the following. Given a continuous function $g \in C(X)$ and an $\epsilon > 0$, need to produce a function f from $\mathcal{A}$ satisfying

$$g(x) - \epsilon \leq f(x) \leq g(x) + \epsilon$$

for all x. Let a, b be two different points in X. There is function function f that separates them, i.e. $f(a) \neq f(b)$. By taking linear combination with constants, i.e. a function of the form $\alpha f + \beta$ we can match $f(a) = g(a)$ and $f(b) = g(b)$. Let us call this function $f_{ab}(x)$. $f_{ab} \in \mathcal{A}$ and $f_{ab}(a) = g(a), f_{ab}(b) = g(b)$. By continuity for $b \neq a$, there is an open set $N_{a,b}$ containing a, b, and $g(x) - \epsilon \leq f_{ab}(x) \leq g(x) + \epsilon$ on N_{ab}. For fixed a, $\cup_{b: b \neq a} N_{ab} = X$ and there is a finite sub cover with $b \in F$. Let $f_a^*(x) = \wedge_{b \in F} f_{ab}(x)$. Then $f_a^*(x) \leq g(x) + \epsilon$ for all $x \in X$ and

$$g(x) - \epsilon \leq f_a^*(x)$$

on $N_a = \cap_{b \in F} N_{ab}$ which is open and contains a. Since $\{N_a\}$ is an open covering and there is a finite sub cover G

$$f^{**} = \vee_{a \in G} f_a^*$$

will work to give

$$g(x) - \epsilon \leq f^{**}(x) \leq g(x) + \epsilon$$

Compact subsets of $C(X)$

Ascoli-Arzela Theorem. A closed subset $K \subset C(X)$ of continuous functions on a compact metric space X is compact if and only if

$$\sup_{f \in K} \sup_{x \in X} |f(x)| < \infty$$

and

$$\lim_{\delta \to 0} \sup_{f \in K} \sup_{\substack{x,y \\ d(x,y) \leq \delta}} |f(x) - f(y)| = 0$$

Remark. The function

$$\omega_f(\delta) = \sup_{\substack{x,y \\ d(x,y) \leq \delta}} |f(x) - f(y)|$$

is called the modulus of continuity of f and tends to 0 as $\delta \to 0$. It is uniform over any compact set of continuous functions. In fact there is **Dini'sTheorem.**

If X is compact and $f_n(x)$ are continuous functions and $f_n(x) \downarrow 0$ then $f_n \to 0$ uniformly on X.

Proof. Given $\epsilon > 0$ and $x \in X$ there is $n_0(x)$ such that $f_{n_0(x)}(x) < \frac{\epsilon}{2}$. By continuity $f_{n_0(x)}(y) < \epsilon$ in a neighborhood N_x. $\{N_x\}$ is a covering. take a finite subcover $x \in F$. By monotonicity $f_n(x) \leq \epsilon$ for all x and $n \geq \sup_{x \in F} n_0(x)$.

The necessity of Ascoli-Arzela theorem is obvious. $\|f\| = \sup_x |f(x)|$ is continuous and has to be bounded on compact sets. The modulus continuity $\omega_f(\delta)$, tends to 0 monotonically and has to be uniform by Dini's theorem.

Sufficiency. Let D be a countable dense subset of X. Given a bounded sequence from K we can choose by diagonalization a subsequence $f_n(x)$ that converges at every point of D to a limit $f(x)$ defined for $x \in D$. In particular for $x \in D$, as $n, m \to \infty$

$$|f_n(x) - f_m(x)| \to 0$$

Let $\epsilon > 0$ be given. $|f_n(x) - f_n(y)| < \epsilon$ if $d(x, y) < \delta$. Since X is compact there is a finite set F from D such that for any $x \in X$, there is a $y \in F$ with $d(x, y) < \delta$ making $|f_n(x) - f_n(y)| < \epsilon$ for all n. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(x) - f_m(y)| \\ &\leq \sup_{y \in F} |f_n(y) - f_m(y)| + 2\epsilon \end{aligned}$$

Makes f_n Cauchy in $C(X)$.