Metric Spaces. \((X, d)\) is a metric space if \(X\) is provided with a metric \(d : X \times X \to R\) with the following properties.

1. \(d(x, y) = d(y, x)\) for all \(x, y \in X\)
2. \(d(x, y) \geq 0\) if and only if \(x = y\).
3. For \(x, y, z \in X\), \(d(x, z) \leq d(x, y) + d(y, z)\) (Triangle Inequality)

Definition. \(x_n \to x\) in \(X\) if \(d(x_n, x) \to 0\). i.e given any \(\epsilon > 0\) there is \(n_0\) such that \(d(x_n, x) < \epsilon\) for \(n > n_0\).

Definition \(\{x_n\}\) is a Cauchy sequence if \(\lim_{n,m \to \infty} d(x_n, x_m) = 0\). i.e given \(\epsilon\) there is \(n_0\) such that \(d(x_n, x_m) \leq \epsilon\) for \(n, m \geq n_0\).

A convergent sequence is Cauchy. \(d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)\).

If a subsequence \(x_{n_j}\) of a Cauchy sequence of \(x_n\) converges to a limit \(x\) then the entire sequence converges to it.

\[ d(x_n, x) \leq d(x_{n_j}, x) + d(x_{n_j}, x_n) \]

\(X\) is complete if every Cauchy sequence converges to a limit.

Theorem. If \((X, d)\) is not complete, there is a complete space \((Y, D)\) such that there is an embedding \(y = Tx\) of \(X\) into \(Y\) such that \(d(x_1, x_2) = D(Tx_1, Tx_2)\) and \(TX\) is dense in \(Y\). Such a \((Y, D)\) is unique up to isometry, i.e. if \((Y_1, D_1), (Y_2, D_2)\) are two choices then there is a one to one map \(U\) from \(Y_1\) to \(Y_2\) that is onto and \(D_1(y_1, y_2) = D_2(Uy_1, Uy_2)\). \((Y, D)\) is called the completion of \((X, d)\).

Proof. Consider the space \(\mathcal{Z}\) of all Cauchy sequences \(\xi = \{x_n\}\) from \((X, d)\). We define the distance

\[ D(\xi, \eta) = \lim_{n \to \infty} d(x_n, y_n) \]

It is easy to check using triangle inequality that \(|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n + y_m)\). Since \(R\) is complete the Cauchy sequence \(d(x_n, y_n)\) has a limit. Note that this limit is also \(\lim_{n,m \to \infty} d(x_n, y_m)\). It is possible that \(D(\xi, \eta) = 0\). In that case we say they belong to the same equivalence class. The triangle inequality provides transitivity and we have symmetry. \(\mathcal{Z}/\sim\) is taken as the completion of \((Y, D)\) of \((X, d)\). \(X\) is imbedded in \(Y\) by sending \(x\) to the equivalence class of all sequences that converge to \(x\) in particular the equivalence class containing \(\xi_x = \{x, x, x, \ldots, x, \ldots\}\). \(X\) is imbedded densely because if \(\{x_n\}\) is a Cauchy sequence, the equivalence class of Cauchy sequences \(\xi_n\) that that converge to \(x_n\) converges to the equivalence class \(\xi\) containing \(\{x_n\}\).

\[ \lim_{n \to \infty} D(\xi_{x_n}, \xi) = \lim_{n \to \infty} \lim_{m \to \infty} d(x_n, x_m) = 0 \]

Finally we need to prove that \((Y, D)\) is complete. Let \(\{\xi_i\} = \{x_{i,n}\}\) be a Cauchy sequence of Cauchy sequences with \(D(\xi_i, \xi_j) \to 0\) as \(i, j \to \infty\).

\[ \lim_{i,j \to \infty} \lim_{n \to \infty} d(x_{i,n}, x_{j,n}) = 0 \]
For each \( i \) we can choose \( n_i \) such that \( d(x_{i,n}, x_{i,m}) \leq 2^{-i} \) for \( n, m \geq n_i \). Consider the sequence \( x_i = x_{i,n_i} \).

\[
d(x_{i,n}, x_{j,n}) \leq d(x_{i,n}, x_{i,m}) + d(x_{i,m}, x_{j,m}) + d(x_{j,m}, x_{j,n})
\]

We can let \( m \to \infty \).

\[
d(x_{i,n}, x_{j,n}) \leq d(x_{i,n}, x_{i,m}) + d(x_{i,m}, x_{j,m}) + d(x_{j,m}, x_{j,n})
\]

\[
d(x_{i,n}, x_{j,n}) \leq 2^{-i} + D(\xi_i, \xi_j) + 2^{-j}
\]

Makes \( \eta = \{x_{i,n_i}\} \) a Cauchy sequence. We now show that \( D(\xi_k, \eta) \to 0 \) as \( k \to \infty \)

\[
D(\xi_k, \eta) = \lim_{k, \ell \to \infty} d(x_{i,\ell}, x_{k,n_k}) \leq \lim_{k, \ell \to \infty} D(\xi_i, \xi_k) + 2^{-i} + 2^{-k}
\]

and

\[
\lim_{i \to \infty \; \lim_{k, \ell \to \infty}} d(x_{i,\ell}, x_{k,n_k}) = 0
\]

Examples of Metric spaces.

1. \( X = R, \; d(x, y) = |x - y| \)
2. \( X = R^n, \; d(x, y) = \sqrt{\sum (x_i - y_i)^2} \)
3. \( X = R^n, \; d(x, y) = \sum_i |x_i - y_i| \) or \( d(x, y) = \sup_i |x_i - y_i| \)
4. \( X = L_1[0, 1] \). Lebesgue integrable functions \( f(\cdot) \) on \([0, 1] \). \( d(x, y) = \int_0^1 |f(x) - g(x)|dx \)
5. \( X = C[0, 1] \) Continuous functions \( f(\cdot) \) on \([0, 1] \). \( d(x, y) = \int_0^1 |f(x) - g(x)|dx \)
6. \( X = C[0, 1] \) Continuous functions \( f(\cdot) \) on \([0, 1] \). \( d(x, y) = \sup_x |f(x) - g(x)| \)
7. Lebesgue measurable functions \( f(\cdot) \) on \([0, 1] \) such that \( |f(x)|^p \) is integrable. (1 \leq p < \infty).

\[
d(f, g) = \left[ \int_0^1 |f(x) - g(x)|^p dx \right]^\frac{1}{p}
\]

Triangle inequality for \( L_p \). Minkowski Inequality. Holder Inequality. If \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p, q \geq 1 \)

\[
\int |f(x)g(x)|d\mu \leq \left[ \int |f(x)|^p d\mu \right]^{\frac{1}{p}} \left[ \int |g(x)|^q d\mu \right]^{\frac{1}{q}}
\]

\[
\left( \int |f(x)|^p d\mu \right)^{\frac{1}{p}} = \sup_{g: \int |g(x)|^q d\mu \leq 1} \int f(x)g(x) d\mu
\]

\[
\left( \int |f(x) + f_2(x)|^p d\mu \right)^{\frac{1}{p}} = \sup_{g: \int |g(x)|^q d\mu \leq 1} \left[ \int |(f_1(x) + f_2(x))g(x)| d\mu \right]
\]

\[
\leq \sup_{g: \int |g(x)|^q d\mu \leq 1} \int |f_1(x)g(x)| d\mu + \sup_{g: \int |g(x)|^q d\mu \leq 1} \int |f_2(x)g(x)| d\mu
\]

\[
= \left( \int |f_1(x)|^p d\mu \right)^{\frac{1}{p}} + \left( \int |f_2(x)|^p d\mu \right)^{\frac{1}{p}}
\]
Step 1. Let \( x, y \) be nonnegative. \( 1 \leq p, q \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}
\]

Proof. Calculate \( \sup_y [xy - \frac{y^q}{q}] \). Setting the derivative with respect to \( y \) as 0, \( x = y^{q-1} \) or \( y = x^{\frac{q}{p}} = x^{\frac{p}{q}} \)

\[
\sup_y [xy - \frac{y^q}{q}] = x^{1+\frac{p}{q}} - \frac{x^p}{q} = x^p(1 - \frac{1}{q}) = \frac{x^p}{p}.
\]
Proves the inequality.

Step 2.

\[
\int |f(x)g(x)|d\mu \leq \left[ \int |f(x)|^p d\mu \right]^\frac{1}{p} \left[ \int |g(x)|^q d\mu \right]^\frac{1}{q}
\]

Proof. For any \( \lambda > 0 \)

\[
\int |f(x)g(x)|d\mu = \int |(\lambda f(x))(\frac{g(x)}{\lambda})|d\mu \\
\leq \int \left[ \frac{\lambda^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q} \right]d\mu
\]

Minimize over \( \lambda > 0 \). \( \lambda = (\int |g(x)|^q d\mu)^\frac{1}{p+q} (\int |f(x)|^p d\mu)^{-\frac{1}{p+q}} \)

\[
\lambda^p \int |f(x)|^p d\mu = \lambda^{-q} \int |g(x)|^q d\mu = (\int |f(x)|^p d\mu)^\frac{1}{p} (\int |g(x)|^q d\mu)^\frac{1}{q}
\]

and

\[
\frac{1}{p} + \frac{1}{q} = 1
\]

Step 3. Assume that \( \int |f(x)|^p d\mu < \infty \). Clearly

\[
(\int |f(x)|^p d\mu)^\frac{1}{p} \geq \sup_{g: \int |g(x)|^q d\mu \leq 1} \int |f(x)g(x)|d\mu
\]

Take \( g(x) = c(\text{sign}(f(x))|f(x)|^{p-1} \) where

\[
\int |g(x)|^q d\mu = c^q \int |f(x)|^p d\mu = 1
\]

Then

\[
c \int f(x)g(x)d\mu = c \int |f(x)|^p d\mu
\]

and

\[
c = (\int |f(x)|^p d\mu)^{-\frac{1}{q}}
\]

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\[ c \int |f(x)|^p d\mu = (\int |f(x)|^p d\mu)^{1-\frac{1}{q}} = (\int |f(x)|^p d\mu)^\frac{1}{q} \]

It then follows that

\[ (\int |f(x) + g(x)|^p)^\frac{1}{p} \leq (\int |f(x)|^p)^\frac{1}{p} + (\int |g(x)|^p)^\frac{1}{p} \]

**Step 4.** Actually if for a measurable function \( f \)

\[
\sup_{g \in \mathcal{G}} \int_{|g|^q d\mu = 1} f(x)g(x) d\mu < \infty
\]

where \( \mathcal{G} \) consists of functions \( g \) that are bounded and supported on some set \( E \) of finite measure on which \( f \) is bounded as well, then \( \int |f|^p d\mu < \infty \) and the Step 3 is applicable. Replace \( f \) by \( f\chi_E \) and and get a bound for \( \int |f_E(x)|^p d\mu \) that is uniform in \( E \).