Hereafter we will be concerned with a countably additive measure μ on a space X defined for $E \in \Sigma$, a $\sigma-$field of subsets of X . We assume that $\mu(X) < \infty$. In fact $\mu(X) = 1$.

A sequence of measurable functions f_n can converge to f in different senses.

A sequence of bounded measurable functions $f_n(\cdot)$ converges *uniformly* to $f(\cdot)$ if

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$$

A sequence of measurable functions $f_n(\cdot)$ converges *pointwise* to $f(\cdot)$ if for each $x \in X$

$$\lim_{n\to\infty} f_n(x) = f(x)$$

A sequence of measurable functions $f_n(\cdot)$ converges *almost everywhere* to $f(\cdot)$ if

$$\mu[x : \lim_{n\to\infty} f_n(x) = f(x)] = 1$$

i.e. $f_n(x) \to f(x)$ except possibly for x in an exceptional set of measure 0.

A sequence of measurable functions $f_n(\cdot)$ converges *in measure* to $f(\cdot)$ if for any $\epsilon > 0$

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| \geq \epsilon] = 0$$

Uniform convergence $\Rightarrow$ Pointwise convergence $\Rightarrow$ Almost everywhere convergence $\Rightarrow$ Convergence in measure.

Lemma. If $f_n(\cdot) \to f(\cdot)$ in measure then there is a subsequence n_j such that $f_{n_j}(x) \to f(x)$ almost everywhere.

Proof. We use repeatedly the observation that if a_n is nonnegative and $a_n \to 0$ there is a subsequence a_{n_j} such that $\sum_j a_{n_j} < \infty$. By diagonalization we can choose a subsequence such that for every integer $k \geq 1$,

$$\sum_j \mu[x : |f_{n_j}(x) - f(x)| \geq \frac{1}{k}] < \infty$$

Denoting by $E_{j,k}$ the set $[x : |f_{n_j}(x) - f(x)| \geq \frac{1}{k}]$, we have

$$\mu[\cap_\ell \cup_{j\geq\ell} E_{j,k}] \leq \lim_{\ell\to\infty} \sum_{j\geq\ell} \mu[E_{j,k}] = 0$$

which is the same as

$$\mu[x : \limsup_{n\to\infty} |f_{n_j}(x) - f(x)| \geq \frac{1}{k}] = 0$$

for every k.

Problem 1. Find a sequence $f_n(x)$ on $[0,1]$ such that $f_n(\cdot) \to 0$ in measure with respect to the Lebesgue measure but $\limsup_n f_n(x) = 1$ for every x.

We will now consider integrals of unbounded measurable functions. First nonnegative functions. If $f \geq 0$ and measurable we define

$$\int f d\mu = \sup_{\substack{0 \leq g \leq f \\ g \text{ bounded}}} \int g d\mu$$

i.e the supremum of integrals of nonnegative bounded measurable functions dominated by f. Note that if f is bounded then $g = f$ is the best choice and we recover $\int f d\mu$. Of course the supremum need not be finite. If it is finite $f \geq 0$ is said to be integrable. Otherwise not integrable.

Lemma. If $f_1, f_2 \geq 0$, $\int (f_1 + f_2) d\mu = \int f_1 d\mu + \int f_2 d\mu$.

Proof. If $g_1 \leq f_1$ and $g_2 \leq f_2$, then $g_1 + g_2 \leq f_1 + f_2$ and the integral is additive for bounded measurable functions. $g = g_1 + g_2$ is an admissible choice. Conversely if $g \leq f_1 + f_2$ is given on $f_1 + f_2 = 0$ we can take $g_1 = g_2 = 0$ and on $f_1 + f_2 > 0$ we define $g_1 = \frac{gf_1}{f_1+f_2}, g_2 = \frac{gf_2}{f_1+f_2}$ so that $g_1 \leq f_1$ and $g_2 \leq f_2$ and $g_1 + g_2 = g$.

Fatou's Lemma. If $f_n \geq 0$ and $f_n \to f$,then

$$\int f d\mu \leq \liminf_{n \to \infty} \int f_n d\mu$$

Proof.. Let g be bounded and $0 \leq g \leq f$. Then $\min\{g, f_n\} \to \min\{g, f\} = g$. But $\min\{g, f_n\} \leq f_n$. Also $\min\{g, f_n\}$ is uniformly bounded by a bound for g. Therefore for any such g

$$\int g d\mu = \lim_n \int \min\{g, f_n\} d\mu \leq \liminf_{n \to \infty} \int f_n d\mu$$

Taking sup over g,

$$\int f d\mu \leq \liminf_{n \to \infty} \int f_n d\mu$$

Monotone Convergence Theorem. Let $f_n \geq 0$, $f_n \uparrow f$ then $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$. Clearly $0 \leq f \leq g$ implies $\int f d\mu \leq \int g d\mu$. Therefore $\int f_n d\mu \leq \int f d\mu$ and by Fatou's lemma $\liminf_{n \to \infty} \int f_n d\mu \geq \int f d\mu$. Follows that

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu$$

Similarly if $f_n \geq 0$, $f_n \downarrow f$ and if f_n is integrable for some n, say $n = 1$, then $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$. Consider $g_n = f_1 - f_n \uparrow f_1 - f$. Therefore $\int (f_1 - f_n) d\mu \to \int (f_1 - f) d\mu$. But $\int (f_1 - f_n) d\mu + \int f_n d\mu = \int f_1 d\mu$. If $\int f_1 d\mu < \infty$ it follows that $\lim_{n \to \infty} \int f_n d\mu = \int f d\mu$.

If f is not non negative then $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. f is integrable only if both f^+ and f^- are integrable and $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$. Note that f is integrable if and only if $|f|$ is integrable and $\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu$.

Verify that $\int f d\mu = \Lambda(f)$ is linear. i.e if f and g are integrable so is $f+g$ and $\int (f+g)d\mu = \int f d\mu + \int g d\mu$. $f = f^+ - f^-$ and $g = g^+ - g^-$. $|f|, |g|$ are integrable and so $|f+g| \leq |f| + |g|$ is integrable too. $f + g = (f^+ + g^+) - (f^- + g^-) = (f+g)^+ - (f+g)^-$

$$(f^+ + g^+) - (f+g)^+ = (f^- + g^-) - (f+g)^- = h \geq 0$$

We see that

$$\int (f+g)d\mu = \int (f+g)^+ d\mu - \int (f+g)^- d\mu = \int [(f+g)^+ + h]d\mu - \int [(f+g)^- + h]d\mu$$
$$= \int (f^+ + g^+)d\mu - \int (f^- + g^-)d\mu = \int f d\mu + \int g d\mu$$

Dominated Convergence theorem.

If $f_n \to f$ in measure and $\sup_n |f_n(x)| \leq g(x)$ with g integrable, then $\int f_n d\mu \to \int f d\mu$

Proof. $g(x) + f_n(x) \geq 0$ and $g + f_n \to g + f$. By Fatou's lemma, $\int (g+f)d\mu \leq \liminf \int (g+f)d\mu$. Since $\int g d\mu < \infty$ we can subtract it from both sides to conclude

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int f d\mu$$

Since $g(x) - f_n(x) \geq 0$ as well, it works with $-f_n$ replacing f_n.

$$\limsup_{n\to\infty} \int f_n d\mu \leq \int f d\mu$$

Problem 2.

Consider $f_n(x) = n^p x^n (1-x)$ on $[0,1]$. $\lim_{n\to\infty} f_n(x) = 0$ for $x \in [0,1]$. Determine the values of p for which $\int_{[0,1]} f_n(x)dx \to 0$. Are these the same as those for which $\sup_n f_n(x)$ is integrable on $[0,1]$?

Uniform Integrability.

A collection $\mathcal{A}$ of intgrable functions $\{f\}$ is said to be uniformly integrable if

1. $\sup_{f\in A} \int |f| d\mu \leq C < \infty$

2. Given any $\epsilon > 0$ there is a $\delta > 0$ such that for any $E \in \Sigma$ with $\mu(E) < \delta$

$$\sup_{f\in\mathcal{A}} \int_E |f| d\mu < \epsilon$$

Lemma. If f is integrable then for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\mu(E) < \delta$, $\int_E |f(x)| d\mu < \epsilon$.

Proof. Can assume $f \geq 0$. Then $f_n = \min\{f, n\} \uparrow f$ and $\int |f - f_n| d\mu \to 0$. Pick k such that $\int |f - f_k| d\mu < \frac{\epsilon}{2}$. f_k is bounded by k. If $\mu(E) < \frac{\epsilon}{2k}$, then

$$\int_E f d\mu = \int_E (f - f_k) d\mu + \int_E f_k d\mu \leq \int (f - f_k) d\mu + k\mu(E) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Lemma. Let $f_n \geq 0$, and $f_n \to f$ in measure. f_n, f are integrable. Then the following are equivalent.

1. $\int |f_n - f| d\mu \to 0$

2. $\{f_n\}$ are uniformly integrable.

3.$\int f_n d\mu \to \int f d\mu$

Proof. 1 $\Rightarrow$ 2. Let $\epsilon > 0$ be given. Since f is integrable there is $\delta_0 > 0$ such that $\mu(E) < \delta_0$ implies $\int_E f d\mu \leq \frac{\epsilon}{2}$. There is k such that for $n \geq k$, $\int_E |f_n - f| d\mu \leq \frac{\epsilon}{2}$ and therefore for any such E, $\int_E f_n d\mu \leq \epsilon$. Since $f_1, \ldots, f_k$ are integrable for a given ϵ there are $\delta_1, \ldots, \delta_k$ that work. Since k is finite $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_k\}$ works.

2 $\Rightarrow$ 3. We saw earlier that for any $\epsilon > 0$ as $n \to \infty$

$$\mu[x : |f_n(x) - f(x)| \geq \epsilon] = \mu[E_n] \to 0$$

Uniform integrability of f_n and the integrability of f imply that $\int_{E_n} |f_n - f| d\mu \to 0$. The integral over E_n^c is at most $\epsilon\mu(X)$ and $\epsilon > 0$ is arbitrary.

3. $\Rightarrow$ **2**. If not (along a subsequence) there are sets E_n with $\mu[E_n] \to 0$ but $\int_{E_n} f_n d\mu \geq \epsilon > 0$. Let $g_n = f_n(1 - \chi_{E_n})(x)$. Then $g_n \to f$ in measure because it differs from f_n only on E_n and $\mu[E_n] \to 0$. By Fatou's lemma

$$\int f d\mu \leq \liminf_{n\to\infty} \int g_n d\mu = \liminf_{n\to\infty} [\int f_n d\mu - \int_{E_n} f_n d\mu] \leq \int f d\mu - \epsilon$$

A contradiction.

2. $\Rightarrow$ **1**. If $f_n \to f$ in measure then the set

$$\{x : |f_n(x)| \geq k\} \subset \{x : |f_n(x) - f(x)| \geq 1\} \cup \{x : |f(x)| \geq k - 1\}$$

Let $\epsilon > 0$ be given. $\mu[\{x : |f_n(x) - f(x)| \geq 1\}] < \frac{\epsilon}{2}$ for $n \geq n_0$. Choose k such that $\mu\{x : |f(x)| \geq k - 1\}] < \frac{\epsilon}{2}$. Then $\mu[\{x : |f_n(x)| \geq k\}] \leq \epsilon$ for $n \geq n_0$. In other words

$$\lim_{k\to\infty} \sup_n \mu[x : |f_n(x)| \geq k] = 0$$

Replace f_n by $\min\{f_n, k\}$. Use bounded convergence theorem. Use uniform integrability to control the integral of f_n on the set where it is larger than k, which has small measure.