Spectrum of Compact Operators. Let \( T : \mathcal{H} \to \mathcal{H} \) be a compact, self adjoint linear operator. Consider the quadratic form \( Q(x) = \langle Tx, x \rangle \). Let
\[
\sup_{\|x\| \leq 1} Q(x) = \lambda > 0
\]
Then it is attained at some \( y \) with \( \|y\| = 1 \) and \( Ty = \lambda y \). There is a sequence \( x_n \) with \( \|x_n\| \leq 1 \) and \( \langle Tx_n, x_n \rangle \to \lambda \). We can assume by taking a subsequence that \( x_n \to y \) weakly. Then since \( T \) is compact \( T x_n \) will converge to \( Ty \) in norm and consequently
\[
Q(x_n) \to Q(y) = \lambda
\]
\( \lambda \) is the maximum on \( \|x\| = 1 \) at \( y \). For any \( x \) with \( \|x\| = 1 \) and \( x \perp y \), we have
\[
\|y + tx\| = \sqrt{1 + t^2}
\]
\[
Q\left(\frac{y + tx}{\sqrt{1 + t^2}}\right) \leq Q(y)
\]
The derivative with respect to \( t \) at 0 is 0.
\[
\langle Tx, y \rangle + \langle Ty, x \rangle = 0
\]
Since \( T \) is self adjoint \( \langle Ty, x \rangle = 0 \) whenever \( \langle y, x \rangle = 0 \). This forces \( Ty = cy \) for some \( c \) and \( \langle Ty, y \rangle = c = \lambda \). Consider the case
\[
\inf_{\|x\| \leq 1} Q(x) = \lambda < 0
\]
for negative eigenvalues exhaust all of them on both sides. Any two eigenvectors for different eigenvalues are orthogonal. Let \( Tx = \lambda x, Ty = \mu y \) with \( \lambda \neq \mu \). Then
\[
0 = \langle Tx, y \rangle = \langle Ty, x \rangle = (\lambda - \mu) \langle x, y \rangle
\]
Any sequence of orthonormal vectors tends to 0 weakly. \( \sum_j |\langle x, e_j \rangle|^2 \leq \|x\|^2 \). \( Te_j \to 0 \) in norm. If \( e_j \) are eigenvectors then \( Te_j = \lambda_j e_j \) and \( \lambda_j \to 0 \).

Example. In \( L_2[0,1] \) with Lebesue measure
\[
(Tf)(s) = \int_0^1 \min(s, t)f(t)dt
\]
\[
g(s) = (Tf)(s) = \int_0^s tf(t)dt + \int_s^1 sf(t)dt
\]
g(0) = 0 and
\[
g'(s) = sf(s) - sf(s) + \int_s^1 f(t)dt = \int_s^1 f(t)dt
\]
g'(1) = 0 and
\[
g''(s) = -f(s)
\]
Need to solve \( \lambda f''(s) = -f(s) \) with \( f(0) = f'(1) = 0 \).

\[
f(s) = a \cos cs + b \sin cs
\]
where \( c^2 \lambda = 1 \). \( f(0) = a = 0 \). \( f'(1) = bc \cos cs = 0 \) if \( c = (2n + 1)\frac{\pi}{2} \).
Projections. $\mathcal{K} \subset \mathcal{H}$ is a subspace.

$$K^\perp = \cap_{x \in \mathcal{K}} \{ y : \langle x, y \rangle = 0 \}.$$  

$$(K^\perp)^\perp = \mathcal{K}, \quad \mathcal{H} = \mathcal{K} \oplus K^\perp, \quad x = Px + (I - P)x.$$  

$$\inf_{y \in \mathcal{K}} \| x - y \|^2 = \| (x - Px) \|^2 = \| (I - P)x \|^2$$

$$\mathcal{H} = \oplus_j \mathcal{K}_j$$

$\mathcal{K}_i \perp \mathcal{K}_j$ for $i \neq j$ and if $x \perp \mathcal{K}_i$ for all $j$ then $x = 0$.

Spectral Measures.

Let $T$ be a self-adjoint transformation. Then $T$ is nonnegative definite if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

**Theorem.** If $p(t)$ is a polynomial with real coefficients and $T$ is self-adjoint then $p(T)$ is well defined and is self-adjoint. More over if $\|T\| = c$ and $p(t) \geq 0$ on $[-c, c]$, then $p(T)$ is non-negative definite.

**proof.** If $T$ is self adjoint so are all the powers and their linear combinations. If $p(t)$ is a polynomial it can be factored as $[\Pi_i(t - c_i)][\Pi_j[(t - a_j)^2 + b_j^2]]$ in terms of real and complex roots. All the roots in $[-c, c]$ have to be of even multiplicity and complex roots come in conjugate pairs. The polynomial $p(t)$ can be factored as

$$k[\Pi_i;c_i \leq -c(t - c_i)][\Pi_i;c_i >c(d_i - t)][\Pi_j[(t - a_j)^2 + b_j^2]]$$

or

$$k[\Pi_i;c_i \geq 0(t + c + c_i)][\Pi_i;c_i >0(c - t + d_i)][\Pi_j][(t - a_j)^2 + b_j^2]$$

with $k > 0$ and $c_i, d_i \geq 0$. It can be rewritten as a linear combination with positive weights of

$$(t + c)^\alpha(c - t)^\beta(t - a_j)^2^\gamma, 1$$

We know that $cI - T, cI + T, c^2I - T^2$ and $I$ are positive semi definite. If $A$ is any one of them $\langle Ay, y \rangle \geq 0$ and $y$ can be $(c - T)^\alpha(c + T)^\beta(T - a_j)^\gamma x$. Makes $p(T)$ positive semidefinite.


$$\langle p(T)x, x \rangle = \int_c^c p(t)d\mu_{x,x}(t)$$

$\mu$ is a non-negative measure with total mass $\|x\|^2$.

$$\mu_{x,y}(dt) = \frac{1}{2} [\mu_{x+y}(dt) - \mu_{x,x}(dt) - \mu_{y,y}(dt)]$$
We can replace \( p(t) \) by any bounded measurable function.

\[
\langle f(T)x, y \rangle = \int_c f(t) \mu_{x,y}(t)
\]

\( f \) can be \( \chi_E(t) = 1_E(t) \). Then \( \chi_E(T) = \mu(E) \) is a projection onto the eigenspace corresponding to all the eigenvalues in \( E \).

\[
T = \int_{-c}^c t \mu(t)
\]
or

\[
\langle Tx, y \rangle = \int_{-c}^c t \mu_{x,y}(dt)
\]

Compare it to \( T = \sum \lambda_j P_j \). \( \mu(dt) \) is a projection valued measure.

**Fourier Series.**

A complex Hilbert space is a vector space over the field of complex numbers. \( a_1 x_1 + a_2 x_2 \) is defined for \( a_1, a_2 \in \mathbb{C} \). The inner product \( \langle x, y \rangle \) is linear in \( x \) for each \( y \) and has the property \( \langle x, y \rangle = \overline{\langle y, x \rangle} \). \( \langle x, x \rangle \geq 0 \) and \( \sqrt{\langle x, x \rangle} \) is a norm under which \( \mathcal{H} \) is complete. An orthonormal basis is one such that \( \langle e_i, e_j \rangle = \delta_{i,j} \) for all \( i, j \) and the only vector \( x \) with \( \langle x, e_j \rangle = 0 \) for all \( j \) is \( x = 0 \). Every \( x \in \mathcal{H} \) has a representation in terms of an orthonormal basis

\[
x = \sum_{j=-\infty}^{\infty} \langle x, e_j \rangle \ e_j
\]

with \( \sum_{j=-\infty}^{\infty} | \langle x, e_j \rangle |^2 = ||x||^2 \).

The important example is \( \mathcal{H} = L^2[0, 2\pi] \), with Lebesgue measure. \( \{e_j\}, j \in \mathbb{Z} \) given by \( f_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx} \) where \( i = \sqrt{-1} \).\( \)s. Remember that \( e^{iax} = \cos ax + i \sin ax \). One can check that

\[
\langle f_j, f_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_j(x)^* f_k(x) dx = \delta_{j,k}
\]

If you want to stay in the real world you can consider \( \frac{1}{\sqrt{2\pi}} \), and \( \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \) for \( n \geq 1 \). It is easy to check that it is an orthonormal set on \( \mathcal{H} = L^2[0, 2\pi] \). Why is it complete? Need to prove linear combinations are dense in \( \mathcal{H} \). We can identify 0 and \( 2\pi \) and view them as periodic functions on \( [0, 2\pi] \) or functions on the circle \( S \). It is therefore enough to prove that the linear combinations are dense in the uniform topology in the space of continuous functions on \( S \). Linear combinations is an algebra (trig identities). Contains constants. Distinguishes points. Apply Stone-Weierstrass.
Convergence Properties of Fourier Series.

$f(x)$ is a complex valued function on $[0, 2\pi]$ with $\int_0^{2\pi} |f(x)|^2 dx < \infty$

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx} dx$$

One expects

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

1. We saw that $\{e^{inx}\}$ is a complete orthonormal set. Therefore

$$\int_0^{2\pi} |\sum_{-k}^\ell c_n e^{inx} - f(x)|^2 dx \to 0$$

as $k, \ell \to \infty$. If $f$ is smooth on $S$, i.e., with $f$ and some derivatives matching at 0 and $2\pi$ we can integrate by parts and

$$c_n = \frac{(-i)^d}{n^d} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^{(d)}(x)e^{-inx} dx$$

One expects rapid convergence for smooth functions.

2. Let us compute

$$s_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} c_n e^{inx}$$

$$= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_0^{2\pi} e^{-iny} f(y) e^{inx} dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{i(n-x-y)} dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{-iN(x-y)} \left[ \frac{e^{i(2N+1)(x-y)} - 1}{e^{i(x-y)} - 1} \right] dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(y) \frac{\sin(N + \frac{1}{2})(x-y)}{\sin \frac{x-y}{2}} dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x-z) \frac{\sin(N + \frac{1}{2})z}{\sin \frac{z}{2}} dz$$

$$s_N(x) - f(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(x-z) - f(x)}{\sin \frac{z}{2}} (\sin(N + \frac{1}{2})z) dz$$
If \( f(x-z) - f(x) \) is integrable near 0, then by Riemann-Lebesgue lemma the integral goes to zero. \( |f(y) - f(x)| \leq C|x - y|^\alpha \) is enough.

**Riemann Lebesgue Lemma.**

If \( f(x) \) is integrable \( \int_{-\infty}^{\infty} f(x)e^{itx} \, dx \to 0 \) as \( t \to \infty \). If \( f \) is smooth and has compact support integration by parts will do it. \( f \) can be approximated in \( L_1 \) by smooth function \( g \) such that \( \int |f - g| \, dx < \epsilon \). Then \( \int f e^{itx} \, dx - \int g e^{itx} \, dx \leq \int |f - g| \, dx < \epsilon \).

**Fejer Kernel.**

\[
\frac{s_0 + s_1 + \cdots + s_{N-1}}{N} = \int f(y)k_N(x-y) \, dy
\]

where

\[
k_N(x) = \frac{1}{2\pi N \sin \frac{x}{2}} \left[ \sin \frac{x}{2} + \cdots + \sin \left( \frac{N}{2} \right) x \right]
\]

\[
= \frac{1}{2\pi N} \frac{\sin^2 \frac{N x}{2}}{\sin^2 \frac{x}{2}}
\]

\( k_N(x) \geq 0, \int_0^{2\pi} k_N(x) \, dx = 1, \) and \( k_N(x) \to 0 \) if \( x \neq 0 \). Approximate identity

\[
\int k_N(x-y) f(y) \, dy \to f(x)
\]

uniformly for continuous \( f \). In \( L_p \) for \( f \in L_p \).

**Lemma.**

\[
\| \int k(y-x) f(x) \, dx \|_p \leq \| f \|_p
\]

if \( k \geq 0, \int k \, dx = 1 \) on \( S \) or \( R \).