Assignment 9.

**Problem 1.** Denote by $f = \frac{d\alpha}{d\lambda}$. Then since $\alpha(A) \leq \lambda(A)$ for all $A \in \Sigma$, we have $0 \leq f(x) \leq 1$ a.e. Define

$$A = \{x : f(x) = 0\}, \quad B = \{x : 0 < f(x) < 1\}, \quad C = \{x : f(x) = 1\}$$

Since $\alpha(A) = \int_A f(x)d\lambda$ and $f(x) = 0$ a.e. on $A$, clearly $\alpha(A) = 0$. Since $\alpha + \beta = \lambda$ it is clear that $\frac{d\beta}{d\lambda} = g(x) = 1 - \frac{d\alpha}{d\lambda} = 1 - f(x) = 0$ on $C$ and $\beta(C) = 0$. On the other hand on $B$,

$$\frac{d\alpha}{d\lambda} = f(x), \quad \frac{d\beta}{d\lambda} = 1 - f(x)$$

are both non-zero, hence

$$\frac{d\alpha}{d\beta} = \frac{f(x)}{1 - f(x)}, \quad \frac{d\beta}{d\alpha} = \frac{1 - f(x)}{f(x)}$$

which implies that $\alpha \ll \beta$ and $\beta \ll \alpha$.

**Problem 2.** If $f_n(x) \to a$ in $L_2(\alpha)$ then $f_n(x) \to a$ in measure with respect to $\alpha$ and it then has a subsequence $f_{n_j}(x) \to a$ a.e with respect to $\alpha$. That subsequence still converges to $b$ in $L_2(\beta)$ and by a similar argument has a subsequence converging to $b$ a.e $\beta$. We end up with the same subsequence converging to $a$ a.e. $\alpha$, and $b$ a.e. $\beta$. If we denote the subsequence by $g_j$ then with

$$A = \{x : \lim_{j \to \infty} g_j(x) = a\}, \quad B = \{x : \lim_{j \to \infty} g_j(x) = b\}$$

we have $A \cap B = \emptyset$ and $\alpha(A^c) = \beta(B^c) = 0$.

**Problem 2.** This is a simple calculation.

$$\int x_i d\alpha = a$$

$$\int f_n(x) d\alpha = a$$

If $i \neq j$,

$$\int x_i x_j d\alpha = a^2, \quad \int (x_i - a)(x_j - a)d\alpha = 0$$

and

$$\int x_i^2 d\alpha = a, \quad \int (x_i - a)^2 d\alpha = a(1 - a)$$
\[
\int [f_n(x) - a]^2 \, d\alpha = \frac{1}{n^2} \left[ \sum_{i=1}^{n} \int (x_i - a)^2 \, d\alpha + \sum_{i,j:i\neq j} \int (x_i - a)(x_j - a) \, d\alpha \right] \\
= \frac{1}{n^2} [na(1 - a)] \\
= \frac{a(1 - a)}{n} \\
\rightarrow 0 \quad \text{as } n \rightarrow \infty
\]

Similarly \( \int [f_n(x) - b]^2 \, d\beta \rightarrow 0 \). Now use Problem 1.