Problem 1. We have a measure space $(X, \mathcal{F}, \mu)$ which is not a finite set of points. More precisely assume that there is a countable collection of disjoint measurable sets $\{A_j\}$ such that $0 < \mu(A_j) < \infty$ for each $j$. The goal is to prove that in such a case not every bounded linear functional $\Lambda(f)$ defined on $L_\infty(X, \mathcal{F}, \mu)$ is of the form

$$\Lambda(f) = \int_X f(x) \phi(x) d\mu$$

for some $\phi(x) \in L_1(X, \mathcal{F}, \mu)$. This is to be carried out in several steps.

**Step 1.** Find a closed subspace $B \subset L_\infty(X, \mathcal{F}, \mu)$ which is isomorphic to $\ell_\infty$, the space of all bounded sequences $\{a_n\}$ of real numbers with $\|\{a_n\}\| = \sup_{1 \leq n < \infty} |a_n|$. 

**Step 2.** Use Hahn-Banach theorem to construct a linear functional on $\ell_\infty$ satisfying

$$\lim \inf a_n \leq \Lambda(\{a_n\}) \leq \lim \sup a_n$$

**Step 3.** Show that $\Lambda(\{a_n\})$ cannot be of the form

$$\Lambda(\{a_n\}) = \sum_n a_n p_n$$

for some sequence $\{p_n\}$ with $\sum_n |p_n| < \infty$.

**Step 4.** Transplant $\Lambda$ on $B$ and extend to $L_\infty(X, \mathcal{F}, \mu)$ again by Hahn-Banach theorem. Show that the extension cannot be of the form

$$\Lambda(f) = \int_X f(x) \phi(x) d\mu$$

for some $\phi(x) \in L_1(X, \mathcal{F}, \mu)$.

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a non-atomic measure space (i.e. one in which every set $A$ of finite non-zero measure can be divided into two disjoint sets whose measures are equal) which is $\sigma$-finite but NOT finite, i.e. $\mu(X) = \infty$. If $1 \leq p_1 < p_2 < \infty$, show that there are functions $f_1, f_2$ such that

$$f_1 \in L_{p_1}(X, \mathcal{F}, \mu), \notin L_{p_2}(X, \mathcal{F}, \mu)$$

and

$$f_2 \in L_{p_2}(X, \mathcal{F}, \mu), \notin L_{p_1}(X, \mathcal{F}, \mu)$$

What is the situation when $\mu(X) < \infty$?