1. Integration with respect to $\sigma$-finite measures. Let $A_n$ be disjoint measurable sets with $X = \cup A_n$ and $\mu(A_n) < \infty$. One can define integrability of a non-negative measurable $f$ on $X$ in two ways.

(i). $f$ is integrable if

$$\int f \, d\mu = \sum_n \int_{A_n} f \, d\mu < \infty$$

or

(ii). $f$ is integrable if it is integrable on any set $A$ of finite measure and

$$\int f \, d\mu = \sup_{\substack{A \in \mathcal{B} \\mu(A) < \infty}} \int_A f \, d\mu$$

Show that both definitions are equivalent and define the same value for the integral.

2. Prove Fatou’s lemma for $\sigma$-finite measures.