Assignment 8.

\[ F(x) = \sum_{j : x_j \leq x} p_j \]

\[ A_q = \{ x : \limsup_{h \to 0} \frac{F(x + h) - F(x)}{h} > q \} \]

We can exclude from \( A \) the set \( \{ x_j \} \) which is only countable. For each \( x \in A_q \) given any \( \delta > 0 \), there exists \( h < \delta \) such that

\[ F(x + h) - F(x) \geq qh \]

Since \( F(x) \) is right continuous, one can assume that \( x + h \) as well is not one of the discontinuity points \( \{ x_j \} \). The intervals \([x, x + h]\) form a covering of \( A_q \). We can extract a Vitali sub-cover. In other words, given \( \epsilon > 0 \), we have intervals \([x_i, x_i + h_i]\) that are disjoint \( F(x_i + h_i) - F(x_i) \geq qh_i \) and \( \sum_i h_i \geq (\mu(A_q) - \epsilon) \). This implies that

\[ q(\mu(A_q) - \epsilon) \leq \sum_i [F(x_i + h_i) - F(x_i)] \leq s \]

Since \( \epsilon > 0 \) is arbitrary,

\[ \mu(A_q) \leq \frac{s}{q} \]

At the second step, given \( \eta > 0 \) we pick \( N \) such that

\[ \sum_{j = N + 1}^{\infty} p_j \leq \eta \]

remove the big jumps and write \( F = F_1 + F_2 \) where

\[ F_1(x) = \sum_{j \geq N + 1, x_j \leq x} p_j \]

and

\[ F_2(x) = \sum_{j \leq N, x_j \leq x} p_j \]

For \( F_1 \) with many small jumps that add up to at most \( \eta \)

\[ \mu(B_q) \leq \frac{\eta}{q} \]

where

\[ B_q = \{ x : \limsup_{h \to 0} \frac{F_1(x + h) - F_1(x)}{h} > q \} \]
As for $F_2$ which has only finitely many jumps, for any $x$ which is not one of the jump points $\{x_1, \ldots, x_N\}$, $F_2(x + h) = F_2(x)$ if $h$ is so small that $[x, x + h]$ has none of these points. Therefore

$$\lim_{h \to 0} \frac{F_2(x + h) - F_2(x)}{h} = 0$$

Hence $A_q \subset B_q \cup \{x_1, x_2, \ldots, x_N\}$ and

$$\mu(A_q) \leq \mu(B_q) \leq \frac{\eta}{\epsilon}$$

Since $\eta$ can be made as small as we like, for any $q > 0$,

$$\mu(A_q) = 0$$

Assignment 7.

Problem 1. Assume that $\{f_n\}$ is NOT uniformly integrable. Then there exists a subsequence $n_j$ and measurable subsets $A_{n_j}$ of $X$, such that $\mu(A_{n_j}) \to 0$ while

$$\int_{A_{n_j}} f_{n_j}(x) d\mu \geq \delta > 0$$

Let us denote $A_{n_j}$ by $B_j$ and $f_{n_j}$ by $g_j$. $g_j \to f$ in measure. Since $\mu(B_j) \to 0$, it follows that $g_j 1_{B_j} \to f$ in measure as well. From Fatou’s lemma

$$\int f d\mu \leq \liminf_{j \to \infty} \int g_j 1_{B_j} d\mu = \liminf_{j \to \infty} \int \left[ g_j - g_j 1_{B_j} \right] d\mu \leq \lim_{j \to \infty} \int g_j d\mu - \delta$$

contradicting equality in Fatou’s lemma. Since $\{f_n\}$ is now shown to be uniformly integrable and $f$ is integrable it follows that $|f_n - f|$ is uniformly integrable and therefore $\int |f_n - f| d\mu \to 0$.

$\mu$ can be $\sigma$-finite. Let $\phi > 0$ be integrable. If we define $\lambda(A) = \int \phi(x) d\mu$, then $\lambda$ is a finite measure and

$$\int f d\mu = \int f \phi^{-1} d\lambda$$

$f_n \phi^{-1} \to f \phi^{-1}$ a.e. and $\int f_n \phi^{-1} d\lambda \to \int f \phi^{-1} d\lambda$. We conclude that $f_n \phi^{-1}$ is uniformly integrable with respect to $\lambda$ and

$$\int |f_n \phi^{-1} - f \phi^{-1}| d\lambda = \int |f_n - f| \phi^{-1} d\lambda = \int |f_n - f| d\mu \to 0$$

Problem 2i.

$$\|x_n - x\|^2 \leq \langle x_n - x, x_n - x \rangle$$

$$= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle$$

$$= 2[1 - \langle x_n, x \rangle] \to 0$$
Problem 2ii. We can assume with out loss of generality that \( \int_X |f_n|^2d\mu = \int_X |f|^2d\mu = 1 \) and use problem 2i. We need to establish that

\[
\lim_{n \to \infty} \int f_n(x)g(x)d\mu = \int f(x)g(x)d\mu
\]

for all \( g \in L_2(\mu) \). We have it for \( g = 1_A \). take linear combinations and we have it for simple functions. Simple functions are dense in \( L^2(\mu) \). Finally given \( g \in L_2(\mu) \), for any \( \epsilon > 0 \) we can fine a simple function \( s(x) \) such that

\[
\left\| s - g \right\| = \left[ \int |s(x) - g(x)|^2d\mu \right]^{\frac{1}{2}} \leq \epsilon.
\]

Therefore

\[
\limsup_{n \to \infty} \left| \int g(x)[f_n(x) - f(x)]d\mu \right| \leq \limsup_{n \to \infty} \int |g(x) - s(x)||f_n(x) - f(x)|d\mu \\
\leq \limsup_{n \to \infty} \left[ \left\| g - s \right\| \cdot \left\| f_n - f \right\| \right] \\
\leq \limsup_{n \to \infty} \left[ \left\| g - s \right\| \left[ \left\| f_n \right\| + \left\| f \right\| \right] \right] \\
= 2\epsilon
\]

Since \( \epsilon > 0 \) is arbitrary we are done.

Assignment 6.

Problem 1. Step 1. Start whi with a countable collection of disjoint sets \( \{A_j\} \) with positive measure. Then functions of the form

\[
\sum_j a_j 1_{A_j}(x)
\]

provide a 1 – 1 correspondence between \( \{a_j\} \in \ell_\infty \) and \( g(x) = \sum_j a_j 1_{A_j}(x) \) in \( L_\infty(\mu) \).

\[
\text{esssup } |g(x)| = \sup_j |a_j|
\]

Problem 1. Step 2. Consider in \( \ell_\infty \) the subspace

\[
E = \left[ \xi = \{a_n\} : \Lambda(\xi) = \lim_{n \to \infty} a_n \text{ exists} \right]
\]

\( E \) is a closed subspace of \( \ell_\infty \) and \( \Lambda(\xi) \) is a bounded linear functional on \( E \). By Hahn-banach theorem it can be extended to all of \( \ell_\infty \).
Problem 1. Step 3. Suppose for some \( \{p_j\} \in \ell_1 \),

\[
\Lambda(\xi) = \sum_j a_j p_j
\]

Then if we take \( \xi_n = \{0, \ldots, 0, 1, 1, \ldots\} \), i.e. \( n \) zeros followed by ones, \( \Lambda(\xi_n) = 1 \) for all \( n \).

But for \( \{p_j\} \) in \( \ell_1 \) one cannot have

\[
1 = \sum_{j=n+1}^\infty p_j
\]

for all \( n \).

Problem 1. Step 4. We extend the linear functional to \( L_\infty(\mu) \) from the subspace of functions of the form

\[
\sum_j a_j 1_{A_j}(x)
\]

If

\[
\Lambda(g) = \int g(x)\phi(x)d\mu
\]

for some \( \phi \in L_1(\mu) \), then for any

\[
g(x) = \sum_j a_j 1_{A_j}(x)
\]

\[
\lambda(g) = \int [\sum_j a_j 1_{A_j}(x)]d\mu = \sum_j a_j \mu(A_j)
\]

where \( \mu(A_j) = \int_{A_j} \phi(x)d\mu \) and \( \{p_j\} = \mu(A_j) \in \ell_1 \), providing a contradiction.

Problem 2. Let \( \mu \) be non-atomic. Then there are sets \( \{A_n\} \) that are disjoint and \( a_n = \mu(A_n) \) satisfies \( 0 < a_n < 2^{-n} \) for large \( n \). Consider the function

\[
g(x) = \sum_n 1_{A_n} c_n
\]

If \( g \) is to be in \( L_{p_1} \) but not in \( L_{p_2} \) with \( p_2 > p_1 \), we need

\[
\sum |c_n|^{p_1} a_n < \infty
\]

as well as

\[
\sum |c_n|^{p_2} a_n = \infty
\]
Take $c_n > 0$ to satisfy

$$c_n^{p_2} = \frac{1}{na_n}$$

Then

$$\sum_n c_n^{p_2} a_n = \sum_n \frac{1}{n} = \infty$$

and

$$\sum_n c_n^{p_1} a_n = \sum_n \frac{1}{n} (na_n)^{1-p_2/p_1} < \infty$$

On the other hand, if $\mu(X)$ is infinite we can find disjoint subsets $A_n$ with $a_n = \mu(A_n) \geq n^2$.

Given $p_1 < p_2$ pick

$$c_n^{p_1} = \frac{1}{na_n}$$

so that

$$\sum_n c_n^{p_1} a_n = \sum_n \frac{1}{n} = \infty$$

But now

$$\sum_n c_n^{p_2} a_n = \sum_n \frac{1}{n} \left(\frac{1}{na_n}\right)^{p_2/p_1-1} < \infty$$

**Assignment 5.**

**Problem 1.** Assume

$$\sum_n \int_{A_n} f \, d\mu < \infty$$

Then for any $A$ with $\mu(A) < \infty$ and $f \geq 0$,

$$\int_A f \, d\mu = \int_{\bigcup_n (A \cap A_n)} f \, d\mu = \sum_n \int_{A \cap A_n} f \, d\mu \leq \sum_n \int_{A_n} f \, d\mu$$

so that

$$\sup_{A: \mu(A) < \infty} \int_A f \, d\mu \leq \sum_n \int_{A_n} f \, d\mu$$

On the other hand

$$\sum_{1 \leq n \leq N} \int_{A_n} f \, d\mu = \int_{\bigcup_{n=1}^N A_n} f \, d\mu \leq \sup_{A: \mu(A) < \infty} \int_A f \, d\mu$$

Letting $N \to \infty$,

$$\sum_1^\infty \int_{A_n} f \, d\mu \leq \sup_{A: \mu(A) < \infty} \int_A f \, d\mu$$
Hence if either one is finite so is the other and both are equal.

**Problem 2.** Let \( f_n \geq 0 \) and \( f_n \to f \) a.e. If \( \mu(A) < \infty \), then from Fatou’s lemma proved for finite measures

\[
\int_A f \, d\mu \leq \liminf_{n \to \infty} \int_A f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu
\]

Since this is true for every set \( A \) with \( \mu(A) < \infty \),

\[
\int f \, d\mu = \sup_{A: \mu(A) < \infty} \int_A f \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu
\]

**Assignment 4.**

**Problem 1i.**

\[
E = \bigcup_k \cap_n \{x : f_n(x) \leq k\}
\]

**Problem 1ii.**

\[
f^*(x) = \limsup_{n \to \infty} f_n(x) = \inf_{k \geq 1} \sup_{n \geq k} f_n(x)
\]

is measurable because

\[
\{x : f^*(x) \geq a\} = \cap_{k \geq 1} \{x : \sup_{n \geq k} f_n(x) \geq a\}
\]

\[
= \cap_{k \geq 1} \cap_m \{x : \sup_{n \geq k} f_n(x) > a - \frac{1}{m}\}
\]

\[
= \cap_{k \geq 1} \cap_m \cup_n \{x : f_n(x) > a - \frac{1}{m}\}
\]

Similarly \( f_*(x) = \liminf_{n \to \infty} f_n(x) = \sup_{k \geq 1} \inf_{n \geq k} f_n(x) \) is measurable and so is the set

\[
\{x : f^*(x) = f_*(x)\}
\]

and the restriction of \( f^* = f_* \) to this set.

**Problem 2.** Here \( \mu \) is a finite measure. If \( f_n \to f \) a.e.

\[
\mu \left[ \cap_{n \geq 1} \cup_{k \geq n} \{x : |f_k(x) - f(x)| \geq \epsilon\} \right] = 0
\]

By countable additivity since

\[
\cup_{k \geq n} \{x : |f_k(x) - f(x)| \geq \epsilon\}
\]
is a decreasing sequence of sets
\[ \mu \left[ \bigcup_{k \geq n} \{ x : |f_k(x) - f(x)| \geq \epsilon \} \right] \to 0 \]
as \( n \to \infty. \) But
\[ \mu \left[ x : |f_n(x) - f(x)| \geq \epsilon \right] \leq \mu \left[ \bigcup_{k \geq n} \{ x : |f_k(x) - f(x)| \geq \epsilon \} \right] \to 0 \]

Assignment 3.

Problem 1i. If \( b > a, \)
\[ F(b) - F(a) = \mu \left[ (-\infty, b] \right] - \mu \left[ (-\infty, a] \right] = \mu \left[ (a, b] \right] \geq 0 \]

Problem 1ii.
\[
\lim_{k \to \infty} F(x + \frac{1}{k}) = \lim_{k \to \infty} \mu \left[ (-\infty, x + \frac{1}{k}] \right] = \lim_{k \to \infty} \mu \left[ \bigcap_{k \geq 1} (-\infty, x + \frac{1}{k}] \right] = \mu \left[ (-\infty, x] \right] = F(x)
\]

Problem 1iii.
\[
\lim_{k \to -\infty} F(k) = \lim_{k \to -\infty} \mu \left[ (-\infty, k] \right] = \lim_{k \to -\infty} \mu \left[ \bigcap_{k \geq 1} (-\infty, k] \right] = \mu \left[ [0] \right] = 0
\]
\[
\lim_{k \to \infty} F(k) = \lim_{k \to \infty} \mu \left[ (-\infty, k] \right] = \lim_{k \to \infty} \mu \left[ \bigcup_{k \geq 1} (-\infty, k] \right] = \mu \left[ [R] \right] = 1
\]

Problem 2. We define for intervals \( (a, b] \) where \( a \) can be \(-\infty\) and \( b \) can be \( \infty, \) \( \mu \left[ (a, b] \right] = F(b) - F(a). \) Of course \( (a, \infty] \) is the same as \( (a, \infty). \) We need to prove that if
\[ (a, b] = \bigcup_j (a_j, b_j] \]
then
\[ F(b) - F(a) = \sum_j \left[ F(b_j) - F(a_j) \right] \]
Since one side is obvious it is only necessary to prove
\[ F(b) - F(a) \leq \sum_j \left[ F(b_j) - F(a_j) \right] \]
Then by the Carathéodory extension theorem we can extend \( \mu \) from the semiring of intervals to the Borel \( \sigma \)-field. Because \( F(x) \to 0 \) as \( x \to -\infty \) we can replace \( a \) by a finite number \( a' \) with \( F(a') - F(a) < \epsilon. \) Similarly we can replace \( b \) if it is \( \infty \) by \( b' \) with \( F(b) - F(b') < \epsilon. \) Using right continuity we can replace \( (a_j, b_j] \) by \( (a_j, b'_j] \) with \( F(b'_j) - F(b_j) \leq \epsilon 2^{-j}. \) We now have
\[ F(b') - F(a') \geq F(b) - F(a) - 2\epsilon \]
and \( [a', b'] \subset (a, b] \) is a closed bounded interval. In addition \( (a_j, b'_j] \supset (a_j, b_j] \) is an open covering of \( [a', b'] \) By Heine-Borel theorem there is a finite sub-cover from \( \{(a_j, b'_j]\} \) and
\[ \sum_j F(b_j) - F(a_j) \geq \sum_j \left[ F(b'_j) - F(a_j) \right] - \epsilon 2^{-j} \geq \left[ F(b') - F(a') \right] - \epsilon \geq F(b) - F(a) - 3\epsilon \]
Since \( \epsilon > 0 \) is arbitrary, countable additivity follows. As for uniqueness \( \mu \) is determined on the semiring and therefore on the field generated by disjoint union of sets from the semiring, i.e. disjoint union of intervals \( (a, b]. \) But if two measures agree on a filed they agree on the \( \sigma \)-field generated by the field.