Test functions.

If we want to show that the Brownian motion in one dimension exits in a finite time from the interval $[1, 1]$, we know that the solution
\[ \frac{1}{2} u_{xx} = -1, \quad u(\pm1) = 0 \]
will give $E[\tau|X(0) = x]$. The solution is of course $(1 - x^2)$. In general it is not necessary to solve the equation explicitly. If we can find a function $u(x)$ such that $(\mathcal{L}u)(x) \geq c > 0$ in a region $G$, then for any starting point $x \in G$ the expected exit time from $G$ is finite i.e.
\[ E[\tau_G|X(0) = x] \leq \frac{2}{c} \sup_x |u(x)| \]
The proof uses Itô’s formula to conclude that
\[ u(x(t)) - \int_0^t (\mathcal{L}u)(x(s))ds \]
is a Martingale. Therefore if $\tau$ is a bounded stopping time such that $\tau \leq \tau_G$, then
\[ E[u(x(\tau)) - u(x) - c\tau|x(0) = x] \geq 0 \]
In particular
\[ E[\tau \wedge t|x(0) = x] \leq \frac{2}{c} \sup_x |u(x)| \]
Since this is true for every $t > 0$ by letting $t \to \infty$ we get our result.

Some times we need methods to conclude that $P[\tau < \infty] = 1$ while $E[\tau]$ may be infinite. If for some $c > 0$, we have a positive bounded function $u$ on $\partial G$ satisfying
\[ (\mathcal{L}u)(x) - cu(x) \geq 0 \text{ for } x \in G \]
then
\[ E[e^{-c\tau_G}|x(0) = x] \geq u(x) \]
In particular $P[\tau < \infty|x(0) = x] > 0$. To show that the probability is actually 1, we need to construct sub-solutions $u_c(x)$ such that $u_c(x) \to 1$ as $c \to 0$. The proof is again by Itô’s formula.
\[ d(e^{-ct}u(x(t))) = (\mathcal{L}u - cu)e^{-ct}dt + dM(t) \]
so that $e^{-ct}u(x(t))$ is a sub-martingale. In particular
\[ E[e^{-c(\tau \wedge t)}u_c(x(\tau \wedge t))] \geq u_c(x) \]
Conversely if we have a super-solution with
\[ (\mathcal{L}u)(x) - cu(x) \leq 0 \text{ for } x \in G \]
with \( u(x) \to \infty \) as \( x \to \partial G \), then \( P[\tau < \infty|x(0) = x] = 0 \). Follows from
\[
E[e^{-c(\tau \wedge t)}u(x(\tau \wedge t))] \leq u(x)
\]
or from the Martingale inequality
\[
\sup_{0 \leq t < \tau} e^{-ct}u(x(t)) < \infty \quad \text{a.e.}
\]
Allation: Non-explosion: If we can construct a function \( u(x) > 0 \) such that \( u(x) \to \infty \) as \( |x| \to \infty \) and
\[
\sup_x \frac{(\mathcal{L}u)(x)}{u(x)} < \infty
\]
then the process cannot explode.
Example: If \( a(x) \leq C|x|^2, |b(x)| \leq C|x| \), then with \( u(x) = 1 + |x|^2 \),
\[
\mathcal{L}u \leq Cu
\]
**Difference approximations to PDE**

One way to numerically solve the heat equation
\[
\frac{1}{2}u + \frac{1}{2}u_{xx} = 0; u(T, x) = f(x)
\]
is to approximate it by difference equations
\[
\frac{1}{\delta} [u((j + 1)\delta, kh) - u(j\delta, kh)] + \frac{1}{2h^2} [u((j + 1)\delta, (k + 1)h) + u((j + 1)\delta, (k - 1)h) - 2u((j + 1)\delta, kh)] = 0
\]
Time \( t \) marches in steps of size \( \delta \) and the space \( x \) is made discrete with a spacing of \( h \). Assuming \( N\delta = T \), with \( u(N\delta, kh) = f(kh) \), we iterate
\[
u(j\delta, kh) = \frac{\delta}{2h^2} [u((j + 1)\delta, (k + 1)h) + u((j + 1)\delta, (k - 1)h) - 2u((j + 1)\delta, kh)]
\]
We can let \( \delta \to 0, h \to 0 \) such that \( \delta \leq h^2 \). Then \( u(j\delta, kh) \) will be an average of \( u((j + 1)\delta, (k \pm 1)h) \) and \( u((j + 1)\delta, kh) \). In particular if \( \delta = h^2 \)
\[
u_h(j\delta, kh) = \frac{1}{2} [u_h((j + 1)\delta, (k + 1)h) + u_h((j + 1)\delta, (k - 1)h)]
\]
The convergence of \( u_h(0, 0) \) to the solution \( u(0, 0) \) of the heat equation given by

\[
    u(0, 0) = \int f(y) \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy
\]

is just the central limit theorem for the binomial distribution. Note that

\[
    u_h(0, 0) = \sum_{r=0}^{N} \binom{N}{r} \frac{1}{2N} f((2r - N)h)
\]

where \( h = \sqrt{\delta} = N^{-\frac{1}{2}} \).

We will give an alternate proof. Assume that \( f \) is smooth and the solution \( u(t, x) \) of the heat equation has enough derivatives in \( t \) and \( x \).

Then consider

\[
    \xi_n^h = u(n\delta, X_n^h)
\]

where \( X_n^h \) is a Markov chain with transition probability

\[
    \pi_h(x, dy) = \frac{1}{2} \delta_{x+h}(dy) + \frac{1}{2} \delta_{x-h}(dy)
\]

It is easily seen that (note \( \delta = h^2 \)),

\[
    E[u(n\delta, X_n^h)|X_{n-1}^h] = \frac{1}{2} [u(n\delta, X_{n-1}^h + h) + u(n\delta, X_{n-1}^h - h)]
\]

\[
    = u(n\delta, X_{n-1}^h) + \frac{h^2}{2} u_{xx}(n\delta, X_{n-1}^h) + o(h^2)
\]

\[
    = u((n-1)\delta, X_{n-1}^h) + \delta u_t(n\delta, X_{n-1}^h) + \frac{h^2}{2} u_{xx}(n\delta, X_{n-1}^h) + o(\delta)
\]

\[
    = u((n-1)\delta, X_{n-1}^h) + o(\delta)
\]

Therefore

\[
    E[f(X_N^h)|X_0^h = x] = E[u(T, x_N^h)|X_0^h = x] = u(0, x) + N o(\delta) = u(0, x) + o(1)
\]