
If we model a stochastic process by

\[ dx(t) = \sqrt{a(x(t))}d\beta(t) + b(x(t))dt ; \quad x(0) = x \]

then we saw that \( x(t) \) is a random process with continuous trajectories. If we have some path dependent payoff functional of the type

\[ F(T, x(\cdot)) = \int_0^T e^{-\lambda s} V(x(s)) + e^{-\lambda T} f(x(T)) \]

the payoff is random, and one is often interested in calculating the expected value of the payoff. More generally the model and the payoff could be explicitly time dependent

\[ dx(t) = \sqrt{a(t, x(t))}d\beta(t) + b(t, x(t))dt ; \quad x(s) = x \]

\[ F(s, T, x(\cdot)) = \int_s^T \exp[- \int_s^t \lambda(\sigma, x(\sigma))d\sigma]V(t, x(t))dt + \exp[- \int_s^T \lambda(\sigma, x(\sigma))d\sigma]f(x(T)) \]

and we wish to calculate

\[ u(s, x) = E \left[ F(s, T, x(\cdot)) \middle| x(s) = x \right] \]

as a function of \((s, x)\). From the Markov property it is clear that for times \( s_1 < s_2 \),

\[ u(s_1, x) = E \left[ \int_{s_1}^{s_2} e^{-\int_{s_1}^{t} \lambda(\sigma, x(\sigma))d\sigma} V(t, x(t))dt + e^{-\int_{s_1}^{s_2} \lambda(\sigma, x(\sigma))d\sigma} u(s_2, x(s_2)) \middle| x(s_1) = x \right] \]

We think of \( s_1 = s \) and \( s_2 = s + \epsilon \). Then

\[ u(s, x) - u(s + \epsilon, x) \]

\[ \simeq E \left[ \epsilon V(s, x(s)) + (1 - \epsilon \lambda(s, x(s)))u(s + \epsilon, x(s + \epsilon)) \middle| x(s) = x \right] \]

\[ \simeq \epsilon [V(s, x) - \lambda(s, x)u(s, x)] + E \left[ u(s + \epsilon, x(s + \epsilon)) - u(s + \epsilon, x) \middle| x(s) = x \right] \]
Note that for any smooth function $u$

$$E\left[u(x(s+\epsilon) - u(x(s)) | x(s) = x\right]
\simeq E\left[u_x(x)(x(s+\epsilon) - x(s)) + \frac{1}{2}u_{xx}(x)(x(s+\epsilon) - x(s))^2 | x(s) = x\right]
\simeq \epsilon [b(s, x)u_x(x) + \frac{a(s, x)}{2}u_{xx}(x)]$$

Therefore

$$u(s, x) - u(s + \epsilon, x) = \epsilon[V(s, x) - \lambda(s, x)u(s, x) + b(s, x)u_x(s, x) + \frac{a(s, x)}{2}u_{xx}(s, x)]$$

Or

$$(1) \quad u_s(s, x) + b(s, x)u_x(s, x) + \frac{a(s, x)}{2}u_{xx}(s, x) - \lambda(s, x)u(s, x) + V(s, x) = 0; u(T, x) = f(x)$$

Equations of the above type are called backward parabolic equations. The **MAXIMUM PRINCIPLE** states that if

1. $u, u_x, u_{xx}$ are bounded and continuous and $u$ satisfies equation (1) in $[0, T] \times R$,
2. $b$ and $a$ are bounded and continuous and $a \geq 0$,
3. $V(s, x) \geq 0$ for all $(s, x)$ and $f(x) \geq 0$ for all $x$,

then $u(s, x) \geq 0$ for all $(s, x) \in [0, T] \times R$.

**Proof of the maximum principle.** The basic idea is to show that solutions of (1) cannot achieve their minimum except when $s = T$. Since $u(T, x) = f(x) \geq 0$ this will imply that $\min u(s, x) \geq 0$ and we are done. If the minimum is attained at some $(s_0, x_0)$ with $s_0 < T$, then at that point $u_s(s_0, x_0) \geq 0$ and $u_x(s_0, x_0) = 0$. Moreover $u_{xx}(s_0, x_0) \geq 0$. If only the inequality $\lambda(s_0, x_0) \geq 1$ was true, we would be done. All the terms in the equation are nonnegative and they add up to 0. Since $\lambda(s_0, x_0) \geq 1$, we must have $u(s_0, x_0) \geq 0$. It is easy to achieve $\lambda(s, x) \geq 1$ with out changing the problem. Instead of $u$ we consider the function $v(s, x) = u(s, x)e^{C(s-T)}$ with a constant $C$ to be chosen later. Then $v$ will satisfy

$$v_s(s, x) + b(s, x)v_x(s, x) + \frac{a(s, x)}{2}v_{xx}(s, x) - [C + \lambda(s, x)]v(s, x) + V(s, x)e^{C(s-T)} = 0$$

$v(T, x) = f(x)$
If we pick \( C \) large enough, the new \( \lambda \) which is \( C + \lambda \), can be assumed to be larger than 1. The new \( V \) which is \( V(s, x)e^{C(s-T)} \) is nonnegative since the old one was. Now we will be able to conclude that at the new \( (s_0, x_0) \) where \( v \) achieves the minimum we must have \( v(s_0, x_0) \geq 0 \) and therefore \( v(s, x) \geq 0 \) for all \( (s, x) \). This will imply that \( u(s, x) \geq 0 \) as well. This proof still needs to be fixed. Since \( x \) varies over an unbounded set the infimum may not be attained. We replace \( u(t, x) \) by a new \( v(t, x) \) where

\[
v(t, x) = u(t, x)e^{-eh(x)+C(s-T)}
\]

Think of \( h(x) \) as \( \sqrt{1 + x^2} \). The function \( v \) vanishes as \( |x| \to \infty \) and if it is not nonnegative must now necessarily achieve its negative minimum at some point \( (s_0, x_0) \) with \( s_0 < T \). At this point \( u(s_0, x_0) < 0 \), \( v(s_0, x_0) < 0 \), \( v_s(s_0, x_0) \geq 0 \), \( v_x(s_0, x_0) = 0 \) and \( v_{xx}(s_0, x_0) \geq 0 \). Since

\[
u_s(s_0, x_0) = [v_s(s_0, x_0) - Cv(s_0, x_0)]e^{eh(x_0)-C(s_0-T)} \geq -Cu(s_0, x_0)
\]

\[
u_x(s_0, x_0) = [v_x(s_0, x_0) + eh'(x)v(s_0, x_0)]e^{eh(x_0)-C(s_0-T)} = eh'(x)u(s_0, x_0)
\]

\[
u_{xx}(s_0, x_0) = [v_{xx}(s_0, x_0) + 2eh'(x)v_x(s_0, x_0) + eh''(x)v(s_0, x_0) + e^2[|h'(x)|^2v(s_0, x_0)]]
\]

\[
\geq B[\epsilon + e^2]u(s_0, x_0)
\]

where \( B \) is an upper bound on \( |h'(x)| \) and \( |h''(x)| \). Finally substituting in equation (1)

\[
0 = u_s(s, x) + b(s, x)u_x(s, x) + \frac{a(s, x)}{2}u_{xx}(s, x) - \lambda(s, x)u(s, x) + V(s, x)
\]

\[
\geq -[C + \lambda(s_0, x_0)]u(s_0, x_0) + K[B\epsilon + \frac{B}{2}e^2]u(s_0, x_0)
\]

where \( K \) is an upper bound on \( |b(s, x)| \) and \( a(s, x) \). If \( C + \lambda(s, x) \geq 1 \), for \( \epsilon \) small enough \( K[B\epsilon + \frac{B}{2}e^2] \leq \frac{1}{2} \). Proving that \( u(s_0, x_0) \geq 0 \). This implies that \( v(s_0, x_0) \geq 0 \) which in turn implies \( v(s, x) \geq 0 \) for all \( (s, x) \) and \( u(s, x) \geq 0 \) for all \( (s, x) \)

The maximum principle in particular implies uniqueness. If for given \( a, b, \lambda, V \) and \( f \) we have two solutions \( u \) and \( v \), the difference \( w = u - v \) will be a solution for the same \( a, b \) and \( \lambda \) but with \( V \equiv f \equiv 0 \). It now follows from the maximum principle that \( w \) and \(-w \) are nonnegative. Hence \( w = 0 \) or \( u = v \).

Actually one can use the theory of stochastic differential equations to provide a more direct proof of the maximum principle. Let us suppose that \( u \) is a bounded continuous function, on \([0, T] \times R \) with enough derivatives (two in \( x \) and one in \( s \)), that satisfies

\[
u_s + b(s, x)u_x + \frac{a(s, x)}{2}u_{xx} - \lambda(s, x)u + V(s, x) = 0
\]

Then the function

\[
F(t) = u(t, x(t))e^{-\int_0^t \lambda(s, x(s))\,ds}
\]
where $x(t)$ is a solution of
\[dx(t) = \sqrt{a(t, x(t))}d\beta(t) + b(t, x(t))dt\]
will satisfy
\[
dF(t) = \left[-\lambda(t, x(t))u(t, x(t))dt + u_x dx(t) + u_t dt + \frac{1}{2}u_{xx}(dx(t))^2\right]e^{-\int_0^t \lambda(s, x(s))ds}
\]
\[
= \left[-\lambda u dt + bu_x dt + \sqrt{a}d\beta(t) + u_t dt + \frac{a}{2}u_{xx} dt\right]e^{-\int_0^t \lambda(s, x(s))ds}
\]
\[
- V(t, x(t))e^{-\int_0^t \lambda(s, x(s))ds} dt + e^{-\int_0^t \lambda(s, x(s))ds} \sqrt{a(t, x(t))}d\beta(t)
\]
Or
\[
F(T) - F(0) + \int_0^T V(t, x(t))e^{-\int_0^t \lambda(s, x(s))ds} dt + \int_0^T e(s) d\beta(s)
\]
for some $e$. In particular this has mean zero. $F(0)$ is a constant and equals $u(0, x)$. Hence
\[
u(0, x) = E\left[F(T) + \int_0^T V(t, x(t))e^{-\int_0^t \lambda(s, x(s))ds} dt | x(0) = x\right]
\]
\[
= E\left[f(x(T))e^{-\int_0^T \lambda(s, x(s))ds} + \int_0^T V(t, x(t))e^{-\int_0^t \lambda(s, x(s))ds} dt | x(0) = x\right]
\]
The above relationship between the solution $u$ of a PDE and expectations of certain path dependent functions of solutions $x(\cdot)$ of an SDE is a crucial link between the two. We provided a proof based essentially on Itô’s formula that computed
\[
dF(t) = h(t)dt + H(t)d\beta(t)
\]
and because any integral $\int_{t_1}^{t_2} H(s)d\beta(s)$ had expectation 0, we concluded that
\[
E\left[F(t_2) - F(t_1) - \int_{t_1}^{t_2} h(s)ds | x(t_1) = x\right] = 0
\]
for any $t_1 < t_2$ and $x$. By the Markov property this implies that
\[
E\left[F(t_2) - F(t_1) - \int_{t_1}^{t_2} h(s)ds | x(t) = x\right] = 0
\]
so long as $t \leq t_1 < t_2$. We can avoid the explicit use of Itô’s formula if we want. Consider the quantity
\[
k(t) = E\left[u(t, x(t))e^{-\int_0^t \lambda(s, x(s))ds} + \int_0^t V(s, x(s))e^{-\int_0^s \lambda(\sigma, x(\sigma))d\sigma} | x(0) = x\right]
\]
We will show that $k(t)$ is a constant as a function of $t$. In particular

$$u(0, x) = k(0) = k(T)$$

$$= E\left[f(x(T))e^{-\int_0^T \lambda(s, x(s)) ds} + \int_0^T V(s, x(s))e^{-\int_0^t \lambda(s, x(s))ds} ds | x(0) = x\right]$$

It is clearly sufficient to calculate $k'(t)$ and show that it is identically 0. We will estimate $k(t + h) - k(t)$ and see why this is $o(h)$ due to cancellations and not $O(h)$. It is enough to show

$$E\left[(u(t + h, x(t + h)) - u(t, x(t)))e^{-\int_0^t \lambda(s, x(s)) ds} | x(0) = x\right]$$

$$+ E\left[u(t, x(t))(e^{-\int_t^{t+h} \lambda(s, x(s)) ds} - 1) | x(0) = x\right]$$

$$+ E\left[\int_t^{t+h} V(s, x(s))e^{-\int_t^s \lambda(s, x(s))ds} ds | x(0) = x\right] = o(h)$$

We first condition the path $x(s)$ upto time $t$. This gives a common factor of $e^{-\int_0^t \lambda(s, x(s)) ds}$ that can be pulled out leaving for us to show that

$$E\left[(u(t + h, x(t + h)) - u(t, x(t))) | x(t) = x\right]$$

$$+ E\left[u(t, x(t))(e^{-\int_t^{t+h} \lambda(s, x(s)) ds} - 1) | x(t) = x\right]$$

$$+ E\left[\int_t^{t+h} V(s, x(s))e^{-\int_t^s \lambda(s, x(s))ds} ds | x(t) = x\right] = o(h)$$

The second and third term are easy. They yield

$$[-u(t, x(t))\lambda(t, x(t)) + V(t, x(t))]h + o(h)$$

The first term has to be expanded by Taylor’s formula

$$u(t + h, x(t + h)) - u(t, x(t)) = u_t(t, x)h + u_x(t, x)E[(x(t + h) - x(t)) | x(t) = x]$$

$$+ \frac{1}{2}u_{xx}(t, x)E[(x(t + h) - x(t))^2 | x(t) = x]$$

$$= h[u_t(t, x)] + b(t, x)u_x(t, x) + \frac{1}{2}a(t, x)u_{xx}(t, x) + o(h)$$

Since $u$ satisfies the equation

$$u_t(t, x) + b(t, x)u_x(t, x) + \frac{1}{2}a(t, x)u_{xx}(t, x) - \lambda(t, x)u(t, x) + V(t, x) = 0$$

all the $O(h)$ terms cancel out to give $k'(t) \equiv 0$. 

5
The representation of the solution as an expectation shows that $u$ is nonnegative if $f$ and $V$ are. It also proves uniqueness. In particular if we pick $V \equiv \lambda \equiv 0$, then the solution of
\[ dx(t) = \sqrt{a(t, x(t))} \, d\beta(t) + b(t, x(t)) \, dt : x(s) = x \]
and
\[ u_s(s, x) + b(s, x)u_x(s, x) + \frac{a(s, x)}{2} u_{xx}(s, x) = 0 ; u(T, x) = f(x) \]
are related by
\[ u(s, x) = E[f(x(T)) \mid x(s) = x] \]
If we denote by $p(s, x, t, y) \, dy$ the transition probability density $P[x(t) \in dy \mid x(s) = x]$ then
\[ u(s, x) = \int f(y) p(s, x, T, y) \, dy \]
One can use results from PDE that tell us that the equation (2) has a nice solution and in fact the solution is given by a formula of the type (3). The densities $p(s, x, t, y)$ satisfy
\[ p(s, x, t, y) \geq 0 ; \int p(s, x, t, y) \, dy = 1 \]
and for $s < \sigma < t$
\[ \int p(s, x, \sigma, z) p(\sigma, z, t, y) \, dz = p(s, x, t, y) \]
One can then construct directly a Markov process with transition probabilities $\{p(s, x, t, y)\}$ This will be statistically the same as the solution of the stochastic differential equation.

Examples:

1. Consider the SDE
\[ dx(t) = \, dt + d\beta(t) ; x(s) = x \]
with a solution $x(t) = x + \beta(t) - \beta(s) + t - s$. The probability density of $x(t)$ is Gaussian with mean $x + t - s$ and variance $t - s$ and is given by
\[ p(s, x, t, y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x-t+s)^2}{2(t-s)}} \]
One can check that for fixed $t$ and $y$, $p$ satisfies
\[ p_s + \frac{1}{2} p_{xx} + p_x = 0 \]
2*. For the linear stochastic differential equation
\[ dx(t) = bx(t) \, dt + \sigma x(t) \, d\beta(t) \]
calculate explicitly the probability density \( p(s, x, t, y) = p(t-s, x, y) \) by solving the stochastic differential equation. For any \( \alpha \in R \) calculate the integral

\[
u_{\alpha}(t, x) = \int_0^{\infty} y^{\alpha} p(t, x, y) dy
\]

Verify that \( u_{\alpha} \) satisfies the equation

\[
u_t = b x u_x + \frac{\sigma^2 x^2}{2} u_{xx}
\]

3*. Solve explicitly the stochastic differential equation

\[dx(t) = -\alpha x(t) dt + d\beta(t); x(0) = x\]

Find \( p(t, x, y) \). Show that for each fixed \( y \), \( p(t, x, y) \) satisfies the equation

\[
u_t = -\alpha x p_x + \frac{1}{2} p_{xx}
\]

The functions \( p(s, x, t, y) \) or in the time homogeneous case, \( p(t-s, x, y) \) satisfy the equation

\[
u_p(s, x) + b(s, x) p_x(s, x) + \frac{a(s, x)}{2} p_{xx}(s, x) = 0
\]

In particular any integral \( \int p(s, x, t, y) f(y) dy \) will satisfy

\[
u_u(s, x) + b(s, x) u_x + \frac{a(s, x)}{2} u_{xx}(s, x) = 0
\]

with

\[
\int f(y)p(s, x, t, y) dy \to f(x)
\]

as \( s \to t \). In PDE they are called fundamental solutions and yield directly the transition probabilities.