8. Stochastic Differential Equations as limits of Markov Chains

Instead of a random walk which has increments or steps whose distributions are independent of their current position, we can have Markov Chains moving in $R$, that take small steps, but the distribution of the steps depend on the current position of the Markov Chain. We think of a small parameter $h > 0$ as the unit of time and the Markov Chain from the position $X_n^h$ at time $nh$ moves to its next position $X_{n+1}^h$ with a step of or increment of $Y_{n,n+1}^h = X_{n+1}^h - X_n^h$. We anticipate that in the limit as $h \to 0$ only the mean and variance of the increment $Y_{n,n+1}^h$ will matter. Assuming the transition probabilities to be stationary in time, we denote by

\[
\begin{align*}
  b^h(x) &= E[Y_{n,n+1}^h | X_n^h = x] \\
  a^h(x) &= E[(Y_{n,n+1}^h)^2 | X_n^h = x] \\
  \Delta^h(x) &= E[(Y_{n,n+1}^h)^3 | X_n^h = x]
\end{align*}
\]

We saw earlier that if $b^h(x) = o(h)$, $a^h(x) = h + o(h)$ and $\Delta^h(x) = o(h)$, then the distribution of $X_n^h$ converges to a Normal distribution with mean $X_0 = x$ and variance $t$, provided $nh \to t$. One can improve this to the convergence of $X_n^h$ to the Brownian Motion $x + \beta(t)$, in the sense that the joint distributions of $\{X_n^h : 1 \leq i \leq k\}$ converges to the joint distributions of $\{x + \beta(t_i) : 1 \leq i \leq k\}$ provided $nh \to t_i$ for $i = 1, \cdots, k$. We will now assume that

\[
\begin{align*}
  b^h(x) &= hb(x) + o(h) \\
  a^h(x) &= ha(x) + o(h) \\
  \Delta^h(x) &= o(h)
\end{align*}
\]

Although $a^h(x)$ is only the second moment and not the variance, the difference which is the square of the mean is $(b^h(x))^2$ and is $O(h^2) = o(h)$ and can be ignored. One way to model such a situation (by no means unique) is to assume

\[
X_{n+1}^h = X_n^h + hb(x) + \sqrt{a(x)} \sqrt{h} \xi_n
\]

where $\{\xi_n\}$ i.i.d. standard normals. Or one can replace $\sqrt{h} \xi_n$ by $\beta((n+1)h) - \beta(nh)$ to get

\[
X_{n+1}^h = X_n^h + hb(X_n^h) + \sqrt{a(X_n^h)} \beta((n+1)h) - \beta(nh))
\]

We can take a formal limit here to arrive at

\[
dX(t) = b(X(t)) dt + \sqrt{a(X(t))} d\beta(t)
\]

This equation cannot be treated as a standard ODE. $\beta(t)$ as we saw is not of bounded variation and even in the integrated form

\[
(1) \quad X(t) = X(0) + \int_0^t b(X(s)) ds + \int_0^t \sqrt{a(X(s))} d\beta(s)
\]

does not make sense at the first glance.
There is a theory developed by K. Itô that treats this. The main ideas are the following steps the details of which we will not go into.

**Step 1.** Approximate integrals of the form \( \int_0^t F(s) d\beta(s) \) by

\[
\mathcal{I}_n = \sum_{j=0}^{n-1} F(t_j) \left[ \beta(t_{j+1}) - \beta(t_j) \right]: 0 = t_0 < t_1 < \cdots < t_n = t
\]

sticking the increments always in the future. If \( F(s) \) only depends on the past history up to time \( t \) then \( F(t_j) \) is independent of \( \beta(t_{j+1}) - \beta(t_j) \) and a simple calculation yields

\[
E[\mathcal{I}_n] = 0
\]

\[
E[\mathcal{I}_n^2] = \sum_{j=0}^{n-1} E[F^2(t_j)(t_{j+1} - t_j)]
\]

suggesting a definition of

\[
\mathcal{I} = \int_0^t F(s) d\beta(s)
\]

for random functions \( F(s) \) that depend only on past history such that

\[
E[\mathcal{I}] = 0
\]

\[
E[\mathcal{I}^2] = E\left[ \int_0^t F^2(s) ds \right]
\]

**Step 2.** Define iteratively

\[
X_{n+1}(t) = x + \int_0^t b(X_n(s)) ds + \int_0^t \sqrt{a(X_n(s))} d\beta(s)
\]

**Step 3.** Using the above iteration, similar to Picard iteration for ODE, prove that \( X_n(\cdot) \) has a limit \( X(\cdot) \), that satisfies the equation (1). Prove uniqueness. One makes the assumption that \( b(x) \) and \( \sqrt{a(x)} \) satisfy the Lipshitz condition

\[
|b(x) - b(y)| \leq A|x - y|
\]

\[
|\sqrt{a(x)} - \sqrt{a(y)}| \leq A|x - y|
\]
**Step 4.** Develop a calculus. (Itô Calculus). If we expand

\[ f(\beta(t)) - f(\beta(0)) = \sum_j [f(\beta((j + 1)h)) - f(\beta(jh))] \]

\[ = \sum_j f'(\beta(jh)) [\beta((j + 1)h) - \beta(jh)] \]

\[ + \sum_j \frac{1}{2} f''(\beta(jh)) [\beta((j + 1)h) - \beta(jh)]^2 \]

\[ + \sum_j O(|\beta((j + 1)h) - \beta(jh)|^3) \]

\[ = \int_0^t f'(\beta(s))d\beta(s) + \frac{1}{2} \int_0^t f''(\beta(s))ds \]

We have used the properties that refining an interval \([0,t]\) into finer and finer partitions leads to

\[ \sum [\beta(t_{j+1}) - \beta(t_j)]^2 \to t \]

and

\[ \sum |\beta(t_{j+1}) - \beta(t_j)|^3 = nO(n^{-\frac{3}{2}}) \to 0 \]

Formally \([d\beta(t)]^2 = dt\) and \([d\beta(t)]^k = 0\) for \(k \geq 3\). In Taylor expansion we always keep two terms. Any mixed term \(d\beta dt\) is equal to 0. With this rule one can start with

\[ dX(t) = b(X(t))dt + \sqrt{a(X(t))}d\beta(t) \]

and get

\[ [dX(t)]^2 = a(X(t))dt \]

or

\[ du(t, X(t)) = u_t(t, X(t))dt + u_x(t, X(t))dX(t) + \frac{1}{2} a(X(t))u_{xx}(t, X(t))dt \]

\[ = u_t(t, X(t))dt + u_x(t, X(t))\sqrt{a(X(t))}d\beta(t) + b(X(t))dt \]

\[ + \frac{1}{2} a(X(t))u_{xx}(t, X(t))dt \]

This is to be interpreted as the identity

\[ u(t, X(t)) - u(0, x) = \int_0^t u_x(s, X(s))\sqrt{a(X(s))}d\beta(s) \]

\[ + \int_0^t g(s, X(s))ds \]

with \(g(t, x) = u_t(t, x) + b(x)u_x(t, x) + \frac{1}{2} a(x)u_{xx}(t, x)\).
Examples.

1. Let $f(x) = x^2$. Then

$$\beta(t)^2 - \beta(0)^2 = 2 \int_0^t \beta(s) d\beta(s) + t$$

This can be directly verified by approximation and using the relation

$$\sum_j [\beta(t_{j+1}) - \beta(t_j)]^2 \to t$$

2*. Show that the solution of $dX(t) = X(t) d\beta(t)$ is $X(t) = X(0) \exp[\beta(t) - \frac{t}{2}]$ and not $X(0) \exp[\beta(t)]$.

3*. If $u(t, x)$ satisfies the PDE

$$u_t(t, x) + b(x)u_x(t, x) + \frac{a(x)}{2} u_{xx}(t, x) \equiv 0 \quad \text{for} \quad 0 \leq s \leq T \quad \text{and} \quad u(T, x) = f(x)$$

and $X(t)$ satisfies

$$X(t) = x + \int_0^t b(X(s)) ds + \int_0^t \sqrt{a(X(s))} d\beta(s)$$

then use Itô calculus and the fact that $E[\int_0^T F(s) d\beta(s)] = 0$, to show that

$$u(0, x) = E[f(X(T))]$$

Remark: Technically one needs to know that

$$E[\int_0^T [F(s)]^2 ds] < \infty$$

to define the integral $\int_0^T F(s) d\beta(s)$ and show that it has mean 0 and its variance is equal to $E[\int_0^T [F(s)]^2 ds]$. Although this can be relaxed somewhat in order to define the stochastic integral, the mean of the integral may cease to exist or may exist and be different from 0 if $E[\int_0^T [F(s)]^2 ds] = \infty$. 