Homework Set 5. due Oct 20.

Q1. A subspace \( W \subset V \) is called an invariant subspace of a linear transformation \( A : V \to V \) if \( Ax \in W \) whenever \( x \in W \), i.e. \( A \) maps \( W \) into itself. Show that any linear transformation \( A \) on a vector space \( V \) of dimension \( n \) over the real numbers has an invariant subspace of dimension 1 or 2. If \( n \) is odd, then it has at least one 1 dimensional invariant subspace, i.e an eigen-vector.

Q2. Let \( V \) be a vector space of dimension \( n \) and \( V' \) its dual. Let \( \{ e_i : 1 \leq i \leq n \} \) be a basis of \( V' \). For \( i < j \), \( f_{i,j} = e_i \wedge e_j \) are viewed as antisymmetric bilinear functionals

\[
f_{i,j}(v_1, v_2) = \langle v_1, e_i \rangle \langle v_2, e_j \rangle - \langle v_1, e_j \rangle \langle v_2, e_i \rangle
\]

on \( V \times V \). Show that \( f_{i,j} \) are linearly independent and constitute a basis for the vector space of antisymmetric bilinear functionals on \( V \times V \).