Infinitely Divisible Distributions.

In the study of limit theorems for sums of independent random variables infinitely divisible distributions play a very important role.

Definition 3.1. A distribution μ is said to be *infinitely divisible* if for every positive integer n, μ can be written as the n-fold convolution $(\lambda_n *)^n$ of some other probability distribution λ_n.

Exercise 3.19. Show that the normal distribution with density

$$p(x) = \frac{1}{\sqrt{2\pi}} exp[-\frac{x^2}{2}]$$

is infinitely divisible.

Exercise 3.20. Show that the Poisson distribution with

$$p(n) = \frac{e^{-n}}{n!} \quad \text{for} \quad n \geq 0$$

is infinitely divisible.

Exercise 3.21. Show that any probabilty distribution supported on a finite set $\{x_1, \ldots, x_k\}$ with

$$\mu[\{x_j\}] = p_j$$

and $p_j \geq 0, \sum_{j=1}^k p_j = 1$ is infinitely divisible if and only if it is degenrate, i.e. $\mu[\{x\}] = 1$ for some x.

Exercise 3.22. Show that for any nonnegative finite measure α with total mass a, the distribution

$$e(F) = e^{-a} \sum_{j=1}^{\infty} \frac{(\alpha *)^j}{j!}$$

with characteristic function

$$\widehat{e(F)}(t) = \exp[\int (e^{itx} - 1) d\alpha]$$

is an infinitely divisible distribution.

Exercise 3.23. Show that the convolution of any two infinitely divisible distributions is again infinitely divisible. In particular if μ is infinitely divisible so is any translate $\mu * \delta_a$ for any real a.

We saw in the last section that the asymptotic behavior of $\mu_n * \delta_{A_n}$ can be investigated by means of the asymptotic behavior of $\lambda_n * \delta_{A_n}$ and the characteristic function $\hat{\lambda}_n$ of λ_n has a very special form

$$\begin{aligned}
\hat{\lambda}_n &= \prod_{j=1}^{k_n} \exp[\,\hat{\beta}_{n,j}(t) - 1 + i\,t\,a_{n,j}\,] \\
&= \exp\,[\sum_{j=1}^{k_n} \int [\,e^{i\,t\,x} - 1\,]\,d\beta_{n,j} \,+\, i\,t \sum_{j=1}^{k_n} a_{n,j}\,] \\
&= \exp\,[\int [\,e^{i\,t\,x} - 1\,]\,dM_n \,+\, i\,t\,a_n] \\
&= \exp\,[\int [\,e^{i\,t\,x} - 1 \,-\, i\,t\,\theta(x)\,]\,dM_n \,+\, i\,t\,[\int \theta(x)\,dM_n + a_n\,]] \\
&= \exp\,[\int [\,e^{i\,t\,x} - 1 \,-\, i\,t\,\theta(x)\,]\,dM_n \,+\, i\,t\,b_n\,] \qquad (3.11)
\end{aligned}$$

We can make any reasonable choice for $\theta(\cdot)$ and we will need it to be a bounded continuous function with

$$|\theta(x) - x| \le C|x|^3$$

near 0. Possible choices are $\theta(x) = \frac{x}{1+x^2}$, or $\theta(x) = x$ for $|x| \le 1$ and sign (x) for $|x| \ge 1$. We now investigate when such things will have a weak limit. Convoluting with δ_{A_n} only changes b_n to $b_n + A_n$.

First we note that

$$\hat{\mu}(t) = \exp\,[\int [\,e^{i\,t\,x} \,-\, 1 \,-\, i\,t\,\theta(x)\,]\,dM \,+\, i\,t\,a]$$

is a characteristic function for any measure M with finite total mass. In fact it is the characteristic function of an infinitely divisible probability distribution. It is not necessary that M be a finite measure for μ to make sense. M could be infinite, but in such a way that it is finite on $\{x : |x| \ge \delta\}$ for every $\delta > 0$, and near 0 it integrates x^2 i.e.,

$$M[x : |x| \ge \delta] \;< \infty \qquad \text{for all} \;\; \delta > o \qquad (3.12)$$

$$\int_{|x|\le 1} x^2 dM \;< \infty \qquad (3.13)$$

To see this we remark that

$$\hat{\mu}_\delta(t) = \exp\,[\int_{|x|\geq\delta}[\,e^{i\,t\,x} \,-1\,-\,i\,t\,\theta(x)\,]\,dM\,+\,i\,t\,a]$$

is a characteristic function for each $\delta > 0$ and because

$$|e^{i\,t\,x} \,-1-\,i\,t\,x\,| \leq C_T\,x^2$$

for $|t| \leq T$, $\hat{\mu}_\delta(t) \to \hat{\mu}(t)$ uniformly on bounded intervals where $\hat{\mu}(t)$ is given by the integral

$$\hat{\mu}(t) = \exp\,[\int[\,e^{i\,t\,x} \,-1\,-\,i\,t\,\theta(x)\,]\,dM\,+\,i\,t\,a]$$

which converges absolutely and defines a characteristic function. Let us call measures that satisfy 3.12 and 3.13, that can be expressed in the form

$$\int \frac{x^2}{1+x^2}\,dM < \infty \tag{3.14}$$

admissible *Levy* measures. Since the same argument applies to $\frac{M}{n}$ and $\frac{a}{n}$ instead of M and a, for any admissible *Levy* measure M and real number a, $\hat{\mu}(t)$ is in fact an infinitely divisible characteristic function. As the normal distribution is also an infinitely divisible probability distribution, we arrive at the following

Theorem 3.20. For every admissible Levy measure M, $\sigma^2 > 0$ and real a

$$\hat{\mu}(t) = \exp\,[\int[\,e^{i\,t\,x} \,-1\,-\,i\,t\,\theta(x)\,]\,dM\,+\,i\,t\,a\,-\,\frac{\sigma^2 t^2}{2}]$$

is the characteristic function of an infinitely divisible distribution μ.

We will denote this distribution μ by $\mu = \mathbf{e}\,(M, \sigma^2, a)$. The main theorem of this section is

Theorem 3.21. In order that $\mu_n = \mathbf{e}\,(M_n, \sigma_n^2, a_n)$ may converge to a limit μ it is necessary and sufficient that $\mu = \mathbf{e}\,(M, \sigma^2, a)$ and the following three conditions 3.15 3.16 and 3.17 are satisfied.

For every bounded continuous function f that vanishes in some neighborhood of 0,

$$\lim_{n\to\infty}\int f(x)dM_n = \int f(x)dM \tag{3.15}$$

For some (and therefore for every) $\ell > 0$ such that $\pm\,\ell$ are continuity points for M, i.e., $M\{\pm\,\ell\} = 0$

$$\lim_{n\to\infty}\left[\sigma_n^2 + \int_{-\ell}^{\ell} x^2 dM_n\right] = \left[\sigma^2 + \int_{-\ell}^{\ell} x^2 dM\right] \tag{3.16}$$

$$a_n \to a \text{ as } n \to \infty. \tag{3.17}$$

Proof. Let us prove the suficiency first. Condition 3.15 implies that for every ℓ such that $\pm\,\ell$ are continuity points of M

$$\lim_{n\to\infty}\int_{|x|\geq\ell}[\,e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\,dM_n = \int_{|x|\geq\ell}[\,e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\,dM$$

and because of condition 3.16, it is enough to show that

$$\begin{aligned}\lim_{\ell\to 0}\limsup_{n\to\infty}\Big|\int_{-\ell}^{\ell}[\,e^{i\,t\,x} &- 1 - i\,t\,\theta(x) + \frac{t^2x^2}{2}\,]\,dM_n \\ &- \int_{\ell}^{\ell}[\,e^{i\,t\,x} - 1 - i\,t\,\theta(x) + \frac{t^2x^2}{2}\,]\,dM\Big| \\ &= 0\end{aligned}$$

in order to conclude that

$$\begin{aligned}\lim_{n\to\infty}\Big[&- \frac{\sigma_n^2 t^2}{2} + \int[\,e^{i\,t\,x} - 1 - i\,t\,\theta(x)]\,dM_n\Big] \\ &= \Big[-\frac{\sigma^2t^2}{2} + \int[\,e^{i\,t\,x} - 1 - i\,t\,\theta(x)]\,dM\Big]\end{aligned}$$

This follows from the estimates

$$\Big|e^{i\,t\,x} - 1 - i\,t\,\theta(x) + \frac{t^2x^2}{2}\Big| \leq C_T\,|x|^3$$

and

$$\int_{-\ell}^{\ell}|x|^3 dM_n \leq \ell\int_{-\ell}^{\ell}|x|^2 dM_n$$

Condition 3.17 takes care of the a_n terms.

We now turn to proving the necessity. If μ_n has a weak limit μ then the absolute values of the characteristic functions $|\hat{\mu}_n(t)|$ are all uniformly close to 1 near 0. Since

$$|\hat{\mu}_n(t)| = \exp\left[-\int (1-\cos t\,x)\,dM_n - \frac{\sigma_n^2 t^2}{2}\right]$$

taking logarithms we conclude that

$$\lim_{t\to 0}\sup_n \left[\frac{\sigma_n t^2}{2} + \int (1-\cos t\,x)\,dM_n\right] = 0$$

This implies 3.18, 3.19 and 3.20 below.

For each $\ell > 0$,

$$\sup_n M_n\{x : |x| \geq \ell\} < \infty \tag{3.18}$$

$$\lim_{A\to\infty}\sup_n M_n\{x : |x| \geq A\} = 0 \tag{3.19}$$

For every $0 \leq \ell < \infty$,

$$\sup_n \left[\sigma_n^2 + \int_{-\ell}^{\ell} |x|^2 dM_n\right] < \infty \tag{3.20}$$

We can choose a subsequence of M_n (which we will denote by M_n as well) that 'converges' in the sense that it satisfies conditions 3.15 and 3.16 of the Theorem. Then $\mathbf{e}\,(M_n, \sigma_n^2, 0)$ converges weakly to $\mathbf{e}\,(M, \sigma^2, 0)$. It is not hard to see that for any sequence of probability distributions α_n if both α_n and $\alpha_n * \delta_{a_n}$ converge to limits α and β respectively, then necessarily $\beta = \alpha * \delta_a$ for some a and $a_n \to a$ as $n \to \infty$. In order complete the proof of necessity we need only establish the uniqueness of the representation.

Lemma 3.22. **(Uniqueness).** Suppose $\mu = \mathbf{e}\,(M_1, \sigma_1^2, a_1) = \mathbf{e}\,(M_2, \sigma_2^2, a_2)$, then $M_1 = M_2$, $\sigma_1^2 = \sigma_2^2$ and $a_1 = a_2$.

Proof. Since $\hat{\mu}(t)$ never vanishes by taking logarithms we have

$$\begin{aligned}\psi(t) &= \Big[-\frac{\sigma_1^2 t^2}{2} + \int [\, e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\, dM_1 + i\,t\,a_1\Big] \\ &= \Big[-\frac{\sigma_2^2 t^2}{2} + \int [\, e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\, dM_2 + i\,t\,a_2\Big] \quad (3.21)\end{aligned}$$

We can verify that for any admissible *Levy* measure M

$$\lim_{t\to\infty} \frac{1}{t^2} \int [\, e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\, dM = 0$$

Consequently

$$\lim_{t\to\infty} \frac{\psi(t)}{t^2} = \sigma_1^2 = \sigma_2^2$$

leaving us with

$$\begin{aligned}\psi(t) &= \Big[\int [\, e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\, dM_1 + i\,t\,a_1\Big] \\ &= \Big[\int [\, e^{i\,t\,x} - 1 - i\,t\,\theta(x)\,]\, dM_2 + i\,t\,a_2\Big]\end{aligned}$$

for a different ψ. If we calculate

$$H(s,t) = \frac{\psi(t+s) + \psi(t-s)}{2} - \psi(t)$$

we get

$$\int e^{i\,t\,x}(1 - \cos s\,x) dM_1 = \int e^{i\,t\,x}(1 - \cos s\,x) dM_2$$

for all t and s. Since we can and do assume that $M\{0\} = 0$ for any admissible *Levy* measure M we have $M_1 = M_2$. If we know that $\sigma_1^2 = \sigma_2^2$ and $M_1 = M_2$ it is easy to see that a_1 must equal a_2. Finally

Corollary 3.23. If μ is any infinitely divisible distribution, then for some admissible Levy measure M, $\sigma^2 > 0$ and real number a, $\mu = \mathbf{e}\,(M, \sigma^2, a)$.

Proof. We can write $\mu = \mu_n *^n = \mu_n * \mu_n * \cdots * \mu_n$ with n terms. If we show that $\mu_n \Rightarrow \delta_0$ then the sequence is uniformly infinitesimal and by the earlier theorem on accompanying laws μ will be the limit of some $\lambda_n = \mathbf{e}(M_n, 0, a_n)$ and therefore has to be of the form $\mathbf{e}(M, \sigma^2, a)$ for some choice of admissible *Levy* measure M, $\sigma^2 > 0$ and real a. In a neighborhood around 0, $\hat{\mu}(t)$ is close to 1 and it is easy to check that

$$\hat{\mu}_n(t) = [\hat{\mu}(t)]^{\frac{1}{n}} \to 1$$

as $n \to \infty$ in that neighborhood. As we saw before this implies that $\mu_n \Rightarrow \delta_0$.

Applications.

1. Convergence to the Poisson Distribution. Let $\{X_{n,j} : 1 \le j \le k_n\}$ be k_n independent random variables taking the values 0 or 1 with probabilities $1 - p_{n,j}$ and $p_{n,j}$ respectively. We assume that

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} p_{n,j} = 0$$

which is the uniform infinitesimality condition. We are interested in the limiting distribution of $S_n = \sum_{j=1}^{k_n} X_{n,j}$ as $n \to \infty$. Since we have to center by the mean we can pick any level say $\frac{1}{2}$ for truncation. Then the truncated means are all 0. The accompanying laws are given by $\mathbf{e}(M_n, 0, a_n)$ with $M_n = (\sum p_{n,j})\delta_1$ and $a_n = (\sum p_{n,j})\,\theta(1)$. It is clear that a limit exists if and only if $\lambda_n = \sum p_{n,j}$ has a limit λ as $n \to \infty$ and the limit in such a case is the Poisson distribution with parameter λ.

2. Convergence to the normal distribution. If the limit of $S_n = \sum_j X_{n,j}$ of k_n uniformly infinitesimal mutually independent random variables exists, then the limit is Normal if and only if $M \equiv 0$. If $a_{n,j}$ is the centering needed, this is equivalent to

$$\lim_{n\to\infty} \sum_j P[|X_{n,j} - a_{n,j}| \ge \epsilon] = 0$$

for all $\epsilon > 0$. Since $\lim_{n\to\infty} \sup_j |a_{n,j}| = 0$, this is equivalent to

$$\lim_{n\to\infty} \sum_j P[|X_{n,j}| \ge \epsilon] = 0$$

for each $\epsilon > 0$.

3. The limiting variance and the mean are given by

$$\sigma^2 = \lim_{n\to\infty} \sum_j E\Big\{[X_{n,j} - a_{n,j}]^2 : |X_{n,j} - a_{n,j}| \le 1\Big\}$$

and

$$a = \lim_{n\to\infty} \sum_j a_{n,j}$$

where

$$a_{n,j} = \int_{|x|\le 1} x \, d\alpha_{n,j}$$

Suppose that $E[X_{n,j}] = 0$ for all $1 \le j \le k_n$ and n. Assume that $\sigma_n^2 = \sum_j E\{[X_{n,j}]^2\}$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ exists. What do we need in order to make sure that the limiting distribution is normal with mean 0 and variance σ^2? Let $\alpha_{n,j}$ be the distribution of $X_{n,j}$.

$$|a_{n,j}|^2 = \Big|\int_{|x|\le 1} x \, d\alpha_{n,j}\Big|^2 = \Big|\int_{|x|>1} x \, d\alpha_{n,j}\Big|^2 \le \alpha_{n,j}[\,|x| > 1\,] \int |x|^2 \, d\alpha_{n,j}$$

and

$$\begin{aligned}
\sum_j |a_{n,j}|^2 &\le \Big\{\sum_{1\le j\le k_n} \int |x|^2 \, d\alpha_{n,j}\Big\}\Big\{\sup_{1\le j\le k_n} \alpha_{n,j}[\,|x| > 1\,]\Big\} \\
&\le \sigma_n^2 \Big\{\sup_{1\le j\le k_n} \alpha_{n,j}[\,|x| > 1\,]\Big\} \\
&\to 0
\end{aligned}$$

Because $\sum_j |a_{n,j}|^2 \to 0$ as $n \to \infty$ we must have

$$\sigma^2 = \lim_{n\to\infty} \sum \int_{|x|\le \ell} |x|^2 \, d\alpha_{n,j}$$

for every $\ell > 0$ or equivalently

$$\lim_{n\to\infty} \sum \int_{|x|>\ell} |x|^2 \, d\alpha_{n,j} = 0$$

for every ℓ establishing the necessity as well as sufficiency in Lindeberg's Theorem. A simple calculation shows that

$$\sum_j |a_{n,j}| \le \sum_j \int_{|x|>1} |x|\, d\alpha_{n,j} \le \sum_j \int_{|x|>1} |x|^2\, d\alpha_{n,j} = 0$$

establishing that the limiting Normal distribution has mean 0.

Exercise 3.24. What happens in the Poisson limit theorem (application 1) if $\lambda_n = \sum_j p_{n,j} \to \infty$ as $n \to \infty$? Can you show that the distribution of $\frac{S_n - \lambda_n}{\sqrt{\lambda_n}}$ converges to the standard Normal distribution?

Laws of the iterated logarithm.

When we are dealing with a sequence of independent identically distributed random variables $X_1, \cdots, X_n, \cdots$ with mean 0 and variance 1, we have a strong law of large numbers asserting that

$$P\Big\{\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = 0\Big\} = 1$$

and a central limit theorem asserting that

$$P\Big\{\frac{X_+ \cdots + X_n}{\sqrt{n}} \le a\Big\} \to \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp[-\frac{x^2}{2}]\, dx$$

It is a reasonable question to ask if the random variables $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ themselves converge to some limiting random variable Y that is distributed according to the the standard normal distribution. The answer is no and is not hard to show.

Lemma 3.24. For any sequence n_j of numbers $\to \infty$,

$$P\Big\{\limsup_{j\to\infty} \frac{X_1 + \cdots + X_{n_j}}{\sqrt{n_j}} = +\infty\Big\} = 1$$

Proof. Let us define

$$Z = \limsup_{j\to\infty} \frac{X_1 + \cdots + X_{n_j}}{\sqrt{n_j}}$$

which can be $+\infty$. Because the normal distribution has an infinitely long tail, i.e the probability of exceeding any given value is positive, we must have

$$P[Z \geq a] > 0$$

for any a. But Z is a random variable that does not depend on the particular values of $X_1, \cdots, X_n$ and is therefore a set in the tail σ-field. By Kolmogorov's zero-one law $P[Z \geq a]$ must be either 0 or 1. Since it cannot be 0 it must be 1. The lemma is proved.

Since we know that $\frac{X_1+\cdots+X_n}{n} \to 0$ with probability 1 as $n \to \infty$, the question arises as to the rate at which this happens. The law of the iterated logarithm provides an answer.

Theorem 3.25. For any sequence $X_1, \cdots, X_n, \cdots$ of independent identically distributed random variables with mean 0 and Variance 1,

$$P\left\{ \limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{\sqrt{n \log\log n}} = \sqrt{2} \right\} = 1$$

We will not prove this theorem in the most general case which assumes only the existence of two moments. We will assume instead that $E[|X|^{2+\alpha}] < \infty$ for some $\alpha > 0$. We shall first reduce the proof to an estimate on the tail behavior of the distributions of $\frac{S_n}{\sqrt{n}}$ by a careful application of the Borel-Cantelli Lemma. This estimate is obvious if $X_1, \cdots, X_n, \cdots$ are themselves normally distributed and we will show how to extend it to a large class of distributions that satisfy the additional moment condition. It is clearthat we are interested in showing that for $\lambda > \sqrt{2}$,

$$P\left\{ S_n \geq \lambda\sqrt{n \log\log n} \ \text{ infinitely often } \right\} = 0$$

It would be sufficient because of Borel-Cantelli lemma to show that for any $\lambda > \sqrt{2}$,

$$\sum_n P\left\{ S_n \geq \lambda\sqrt{n \log\log n} \right\} < \infty$$

This however is too strong. The condition of the Borel-Cantelli lemma is not necessary in this context because of the strong dependence between the partial sums S_n. The function $\phi(n) = \sqrt{n \log\log n}$ is clearly well defined and non-decreasing for $n \geq 3$ and it is sufficient for our purposes to show that for any $\lambda > \sqrt{2}$ we can find some sequence $k_n \uparrow \infty$ of integers such that

$$\sum_n P\Big\{\sup_{k_{n-1} \le j \le k_n} S_j \ge \lambda\, \phi(k_{n-1})\Big\} < \infty \tag{3.22}$$

This will establish that with probability 1,

$$\limsup_{n\to\infty} \frac{\sup_{k_{n-1}\le j\le k_n} S_j}{\phi(k_{n-1})} \le \lambda$$

or by the monotonicity of ϕ,

$$\limsup_{n\to\infty} \frac{S_n}{\phi(n)} \le \lambda$$

with probability 1. Since $\lambda > \sqrt{2}$ is arbitrary the upper bound in the law of the iterated logarithm will follow. Each term in the sum of 3.22 can be estimated as in Levy's inequality,

$$P\Big\{\sup_{k_{n-1} \le j \le k_n} S_j \ge \lambda\, \phi(k_{n-1})\Big\} \le 2\, P\Big\{S_{k_n} \ge (\lambda - \sigma)\, \phi(k_{n-1})\Big\}$$

with $0 < \sigma < \lambda$, provided

$$\sup_{1 \le j \le k_n - k_{n-1}} P\Big\{|S_j| \ge \sigma\phi(k_{n-1})\Big\} \le \frac{1}{2}$$

Our choice of k_n will be $k_n = [\rho^n]$ for some $\rho > 1$ and therefore

$$\lim_{n\to\infty} \frac{\phi(k_{n-1})}{\sqrt{k_n}} = \infty$$

and by Chebychev's inequality, for any fixed $\sigma > 0$,

$$\begin{aligned}\sup_{1 \le j \le k_n} P\Big\{|S_j| \ge \sigma\phi(k_{n-1})\Big\} &\le \frac{E[S_n^2]}{[\sigma\phi(k_{n-1})]^2}\\ &= \frac{k_n}{[\sigma\phi(k_{n-1})]^2}\\ &= \frac{k_n}{\sigma^2 k_{n-1} \log\log k_{n-1}}\\ &= o(1) \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.23}$$

By choosing σ small enough so that $\lambda - \sigma > \sqrt{2}$ it is sufficient to show that for any $\lambda' > \sqrt{2}$,

$$\sum_n P\Big\{S_{k_n} \geq \lambda' \, \phi(k_{n-1})\Big\} < \infty$$

By picking ρ sufficiently close to 1, (so that $\lambda' \sqrt{\rho} > \sqrt{2}$), because $\frac{\phi(k_{n-1})}{\phi(k_n)} = \frac{1}{\sqrt{\rho}}$ we can reduce this to the convergence of

$$\sum_n P\Big\{S_{k_n} \geq \lambda \, \phi(k_n)\Big\} < \infty \tag{3.24}$$

for all $\lambda > \sqrt{2}$.

If we use the estimate $P[X \geq a] \leq \exp[-\frac{a^2}{2}]$ that is valid for the standard normal distribution, we can verify 3.24.

$$\sum_n \exp\,[-\frac{\lambda^2(\phi(k_n))^2}{2\,k_n}] < \infty$$

for any $\lambda > \sqrt{2}$.

To prove the lower bound we select again a subsequece, $k_n = [\rho^n]$ with some $\rho > 1$, and look at $Y_n = S_{k_{n+1}} - S_{k_n}$, which are now independent random variables. The tail probability of the Normal distribution has the lower bound

$$\begin{aligned} P[X \geq a] &= \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp[-\frac{x^2}{2}]dx \\ &\geq \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp[-\frac{x^2}{2} - x](x+1)dx \\ &\geq \frac{1}{\sqrt{2\pi}} \exp[-\frac{(a+1)^2}{2}] \end{aligned}$$

If we assume Normal like tail probabilities we can conclude that

$$\sum_n P\Big\{Y_n \geq \lambda\phi(k_{n+1})\Big\} \geq \sum_n \exp\,[-\frac{1}{2}[1 + \frac{\lambda\phi(k_{n+1})}{\sqrt{(\rho^{n+1} - \rho^n)}}]^2] = +\infty$$

provided $\frac{\lambda^2 \rho}{2(\rho-1)} < 1$ and conclude by the Borel-Cantelli lemma, that $Y_n = S_{k_{n+1}} - S_{k_n}$ exceeds $\lambda\phi(k_{n+1})$ infinitely often for such λ. On the other hand from the upper bound we already have (replacing X_i by $-X_i$)

$$P\Big\{\limsup_n \frac{-S_{k_n}}{\phi(k_{n+1})} \le \frac{\sqrt{2}}{\sqrt{\rho}}\Big\} = 1$$

Consequently

$$P\Big\{\limsup_n \frac{S_{k_{n+1}}}{\phi(k_{n+1})} \ge \sqrt{\frac{2(\rho-1)}{\rho}} - \frac{\sqrt{2}}{\sqrt{\rho}}\Big\} = 1$$

and therefore,

$$P\Big\{\limsup_n \frac{S_n}{\phi(n)} \ge \sqrt{\frac{2(\rho-1)}{\rho}} - \frac{\sqrt{2}}{\sqrt{\rho}}\Big\} = 1$$

We now take ρ arbitrarily large and we are done.

We saw that the law of the iterated logarithm depends on two things.
(i). For any $a > 0$ and $p < \frac{a^2}{2}$ an upper bound for the probability

$$P[S_n \ge a\sqrt{n \log\log n}] \le C_p[\log n]^{-p}$$

with some constant C_p
(ii). For any $a > 0$ and $p > \frac{a^2}{2}$ a lower bound for the probability

$$P[S_n \ge a\sqrt{n \log\log n}] \ge C_p[\log n]^{-p}$$

with some, possibly different, constant C_p.

Both inequalities can be obtained from a uniform rate of convergence in the central limit theorem.

$$\sup_a \Big| P\{\frac{S_n}{\sqrt{n}} \ge a\} - \int_a^\infty \frac{1}{\sqrt{2\pi}} \exp[-\frac{x^2}{2}]\, dx \Big| \le C n^{-\delta} \qquad (**)$$

for some $\delta > 0$ in the central limit theorem. Such an error estimate is provided in the following theorem

Theorem 3.26.(Berry-Esseen). Assume that the i.i.d. sequence $\{X_j\}$ with mean zero and variance one satisfies an additional moment condition $E|X|^{2+\alpha} < \infty$ for some $\alpha > 0$. Then for some $\delta > 0$ the estimate $(**)$ holds.

Proof. The proof will be carried out in several lemmas.

Lemma 3.27. Let $-\infty < a < b < \infty$ be given and $0 < h < \frac{b-a}{2}$ be a small positive number. Consider the function $f_{a,b,h}(x)$ defined as

$$f_{a,b,h}(x) = \begin{cases} 0 & \text{for } -\infty < x \le a-h \\ \frac{x-a+h}{2h} & \text{for } a-h \le x \le a+h \\ 1 & \text{for } a+h \le x \le b-h \\ 1-\frac{x-b+h}{2h} & \text{for } b-h \le x \le b+h \\ 0 & \text{for } b+h \le x < \infty \end{cases}$$

For any probability distribution μ with characteristic function $\hat{\mu}(t)$

$$\int_{-\infty}^{\infty} f_{a,b,h}(x)\, d\mu(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\mu}(y) \frac{e^{-i\,a\,y} - e^{-i\,b\,y}}{i\,y} \frac{\sin h\,y}{h\,y}\, dy$$

Proof. This is essentially the Fourier inversion formula. Note that

$$f_{a,b,h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\,x\,y} \frac{e^{-i\,a\,y} - e^{-i\,b\,y}}{i\,y} \frac{\sin h\,y}{h\,y}\, dy$$

We can start with the double integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\,x\,y} \frac{e^{-i\,a\,y} - e^{-i\,b\,y}}{i\,y} \frac{\sin h\,y}{h\,y}\, dy\, d\mu(x)$$

and apply Fubini's theorem to obtain the lemma.

Lemma 3.28. If λ, μ are two probability measures with zero mean having $\hat{\lambda}(\cdot), \hat{\mu}(\cdot)$ for respective characteristic functions. Then

$$\int_{-\infty}^{\infty} f_{a,h}(x)\, d(\lambda - \mu)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\hat{\lambda}(y) - \hat{\mu}(y)] \frac{e^{-i\,a\,y}}{i\,y} \frac{\sin h\,y}{h\,y}\, dy$$

where $f_{a,h}(x) = f_{a,\infty,h}(x)$, is given by

$$f_{a,h}(x) = \begin{cases} 0 & \text{for } -\infty < x \le a-h \\ \frac{x-a+h}{2h} & \text{for } a-h \le x \le a+h \\ 1 & \text{for } a+h \le x < \infty \end{cases}$$

Proof. We just let $b \to \infty$ in the previous lemma. Since $|\hat{\lambda}(y)-\hat{\mu}(y)| = o(|y|)$, there is no problem in applying the Riemann-Lebesgue Lemma. We now proceed with the proof of the theorem.

$$\lambda[[a,\infty)] \le \int f_{a-h,h}(x)\, d\lambda(x) \le \lambda[[a-2h,\infty)]$$

and

$$\mu[[a,\infty)] \le \int f_{a-h,h}(x)\, d\mu(x) \le \mu[[a-2h,\infty)]$$

Therefore if we assume that μ has a density bounded by C,

$$\lambda[[a,\infty)] - \mu[[a,\infty)] \le 2hC + \int f_{a-h,h}(x)\, d(\lambda-\mu)(x)$$

Since we get a similar bound in the other direction as well,

$$\begin{aligned}\sup_a |\lambda[[a,\infty)] - \mu[[a,\infty)]| &\le \sup_a |\int f_{a-h,h}(x)\, d(\lambda-\mu)(x)| \\ &\quad +2hC \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\lambda}(y)-\hat{\mu}(y)| \frac{|\sin h\, y\,|}{h\, y^2}\, dy \\ &\quad +2hC \end{aligned} \tag{3.25}$$

We take λ to be the distribution of $\frac{S_n}{\sqrt{n}}$ having as its characteristic function $\hat{\lambda}_n(y) = [\phi(\frac{y}{\sqrt{n}})]^n$ where $\phi(y)$ is the characteristic function of the common distribution of the $\{X_i\}$ and has the expansion

$$\phi(y) = 1 - \frac{y^2}{2} + O(|y|^{2+\alpha})$$

for some $\alpha > 0$. We therefore get, for some choice of $\alpha > 0$,

$$|\hat{\lambda}_n(y) - \exp[-\frac{y^2}{2}]| \le C\frac{|y|^{2+\alpha}}{n^\alpha} \quad \text{if} \quad |y| \le n^{\frac{\alpha}{2+\alpha}}$$

Therefore for $\theta = \frac{\alpha}{2+\alpha}$

$$\int_{-\infty}^{\infty} |\hat{\lambda}_n(y) \quad - \quad \exp[-\frac{y^2}{2}]| \frac{|\sin h\, y\,|}{h\, y^2}\, dy$$

$$\begin{aligned}
&= \int_{|y|\leq n^\theta} |\hat{\lambda}_n(y) - \exp[-\frac{y^2}{2}]| \frac{|\sin h\, y\,|}{h\, y^2}\, dy \\
&\quad + \int_{|y|\geq n^\theta} |\hat{\lambda}_n(y) - \exp[-\frac{y^2}{2}]| \frac{|\sin h\, y\,|}{h\, y^2}\, dy \\
&\leq \frac{C}{h}\Big\{ \int_{|y|\leq n^\theta} \frac{|y|^\alpha}{n^\alpha}\, dy + \int_{|y|\geq n^\theta} \frac{dy}{|y|^2} \Big\} \\
&\leq C \frac{n^{(\alpha+1)\theta-\alpha} + n^{-\theta}}{h} \\
&= \frac{C}{h n^{\frac{\alpha}{\alpha+2}}}
\end{aligned}$$

Substituting this bound in 3.25 we get

$$\sup_a |\lambda_n[[a,\infty)] - \mu[[a,\infty)]| \leq C_1\, h + \frac{C}{h\, n^{\frac{\alpha}{2+\alpha}}}$$

By picking $h = n^{-\frac{\alpha}{2(2+\alpha)}}$ we get

$$\sup_a |\lambda_n[[a,\infty)] - \mu[[a,\infty)]| \leq C\, n^{-\frac{\alpha}{2(2+\alpha)}}$$

and we are done.