Central Limit Theorem.

We saw before that for any sequence of independent identically distributed random variables $X_1, \cdots, X_n, \cdots$ the sum $S_n = X_1 + \cdots + X_n$ has the property that

$$\lim_{n\to\infty} \frac{S_n}{n} = 0$$

in probability provided the expectation exists and equals 0. If we assume that the Variance of the random variables is finite and equals $\sigma^2 > 0$, then we have

Theorem 3.17. The distribution of $\frac{S_n}{\sqrt{n}}$ converges as $n \to \infty$ to the normal distribution with density

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{x^2}{2\sigma^2}]. \tag{3.1}$$

Proof. If we denote by $\phi(t)$ the characteristic function of any X_i then the characteristic function of $\frac{S_n}{\sqrt{n}}$ is given by

$$\psi_n(t) = [\phi(\frac{t}{\sqrt{n}})]^n$$

We can use the expansion

$$\phi(t) = 1 - \frac{\sigma^2 t^2}{2} + o\,(t^2)$$

to conclude that

$$\phi(\frac{t}{\sqrt{n}}) = 1 - \frac{\sigma^2 t^2}{2n} + o\,(\frac{1}{n})$$

and it then follows that

$$\lim_{n\to\infty} \psi_n(t) = \psi(t) = \exp[-\frac{\sigma^2 t^2}{2}].$$

Since $\psi(t)$ is the characteristic function of the normal distribution with density $p(x)$ given by equation 3.1, we are done.

Exercise 3.14. A more direct proof is possible in some special cases. For instance if each $X_i = \pm 1$ with probability $\frac{1}{2}$, S_n can take the values $n - 2k$ with $0 \le k \le n$,

$$P[S_n = 2k - n] = \frac{1}{2^n}\binom{n}{k}$$

and

$$P[a \le \frac{S_n}{\sqrt{n}} \le b] = \frac{1}{2^n} \sum_{k: a\sqrt{n} \le 2k-n \le b\sqrt{n}} \binom{n}{k}.$$

Use Stirling's approximation to prove directly that

$$\lim_{n\to\infty} P[a \le \frac{S_n}{\sqrt{n}} \le b] = \int_a^b \frac{1}{\sqrt{2\pi}} \exp[-\frac{x^2}{2}]\, dx$$

Actually for the proof of the central limit theorem we do not need the random variables $\{X_j\}$ to have identical distributions. Let us suppose that they all have zero means and that the variance of X_j is σ_j^2. Define $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Assume $s_n^2 \to \infty$ as $n \to \infty$. Then $Y_n = \frac{S_n}{\sqrt{s_n}}$ has zero mean and unit variance. It is not unreasonable to expect that

$$\lim_{n\to\infty} P[Y_n \le a] = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} \exp[-\frac{x^2}{2}]\, dx$$

under certain mild conditions.

Theorem 3.18. **(Lindeberg).** If we denote by α_i the distribution of X_i, the condition

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^n \int_{|x|\ge \epsilon s_n} x^2 d\alpha_i = 0$$

for each $\epsilon > 0$ is sufficient for the central limit theorem to hold.

Proof. The first step in proving this limit theorem as well as other limit theorems that we will prove is to rewrite

$$Y_n = X_{n,1} + X_{n,2} + \cdots + X_{n,k_n} + A_n$$

wher $X_{n,j}$ are k_n mutually independent random variables and A_n is a constant. In our case $k_n = n$, $A_n = 0$, and $X_{n,j} = \frac{X_j}{s_n}$ for $1 \leq j \leq n$. We denote by

$$\phi_{n,j}(t) = E[e^{i\,t\,X_{n,j}}] = \int e^{i\,t\,x} d\alpha_{n,j} = \int e^{i\,t\,\frac{x}{s_n}} d\alpha_j = \phi_j(\frac{t}{s_n})$$

where $\alpha_{n,j}$ is the distribution of $X_{n,j}$. The functions ϕ_j and $\phi_{n,j}$ are the characteristic functions of α_j and $\alpha_{n,j}$ respectively. If we denote by μ_n the distribution of Y_n, its characteristic function $\hat{\mu}_n(t)$ is given by

$$\hat{\mu}_n(t) = \prod_{j=1}^{n} \phi_{n,j}(t)$$

and our goal is to show that

$$\lim_{n\to\infty} \hat{\mu}_n(t) = \exp[-\frac{t^2}{2}]$$

This will be carried out in several steps. First, we define

$$\psi_{n,j}(t) = \exp[\phi_{n,j}(t) - 1]$$

and

$$\psi_n(t) = \prod_{j=1}^{n} \psi_{n,j}(t).$$

We show that for each finite T,

$$\lim_{n\to\infty} \sup_{|t|\leq T} \sup_{1\leq j\leq n} |\phi_{n,j}(t) - 1| = 0$$

and

$$\sup_n \sup_{|t|\leq T} \sum_{j=1}^{n} |\phi_{n,j}(t) - 1| < \infty$$

This would imply that

$$\begin{aligned} \lim_{n\to\infty} \quad & \sup_{|t|\leq T} |\log \hat{\mu}_n(t) - \log \psi_n(t)| \\ \leq \quad & \lim_{n\to\infty} \sup_{|t|\leq T} \sum_{j=1}^{n} |\log \phi_{n,j}(t) - [\phi_{n,j}(t) - 1]| \end{aligned}$$

$$
\begin{aligned}
&\le \quad \lim_{n\to\infty}\sup_{|t|\le T} C\sum_{j=1}^{n}|\phi_{n,j}(t)-1|^2\\
&\le \quad C\lim_{n\to\infty}\Big\{\sup_{|t|\le T}\sup_{1\le j\le n}|\phi_{n,j}(t)-1|\Big\}\Big\{\sup_{|t|\le T}\sum_{j=1}^{n}|\phi_{n,j}(t)-1|\Big\}\\
&= \quad 0
\end{aligned}
$$

by the expansion

$$\log r = \log(1+(r-1)) = r-1+O(r-1)^2.$$

The proof can then be completed by showing

$$\lim_{n\to\infty}\sup_{|t|\le T}|\log\psi_n(t)+\frac{t^2}{2}| = \lim_{n\to\infty}\sup_{|t|\le T}\Big|\Big[\sum_{j=1}^{n}(\phi_{n,j}(t)-1)\Big]+\frac{t^2}{2}\Big| = 0$$

We see that

$$
\begin{aligned}
\sup_{|t|\le T}|\phi_{n,j}(t)-1| &= \sup_{|t|\le T}\Big|\int[\exp[i\,t\,x\,]-1]d\alpha_{n,j}\Big|\\
&= \sup_{|t|\le T}\Big|\int[\exp[i\,t\,\frac{x}{s_n}\,]-1]d\alpha_j\Big|\\
&= \sup_{|t|\le T}\Big|\int[\exp[i\,t\,\frac{x}{s_n}\,]-1-i\,t\,\frac{x}{s_n}]d\alpha_j\Big| \qquad (3.2)\\
&\le C_T\int\frac{x^2}{s_n^2}d\alpha_j \qquad (3.3)\\
&= C_T\int_{|x|<\epsilon s_n}\frac{x^2}{s_n^2}\,d\alpha_j + C_T\int_{|x|\ge\epsilon s_n}\frac{x^2}{s_n^2}\,d\alpha_j\\
&\le C_T\epsilon^2 + C_T\frac{1}{s_n^2}\int_{|x|\ge\epsilon s_n}x^2 d\alpha_j \qquad (3.4)
\end{aligned}
$$

We have used the mean zero condition in deriving equation 3.2 and the estimate $|e^{ix}-1-ix|\le cx^2$ to get to the equation 3.3. If we let $n\to\infty$ by Lindeberg's condition, the second term of equation 3.4 goes to 0. Therefore

$$\limsup_{n\to\infty}\sup_{1\le j\le k_n}\sup_{|t|\le T}|\phi_{n,j}(t)-1|\le\epsilon^2 C_T$$

Since, $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \sup_{|t|\le T} |\phi_{n,j}(t) - 1| = 0$$

Next

$$\sup_{|t|\le T} \sum_{j=1}^{n} |\phi_{n,j}(t) - 1| \le C_T \sum_{j=1}^{n} \int \frac{x^2}{s_n^2} d\alpha_j \le C_T \frac{1}{s_n^2} \sum_{j=1}^{n} \sigma_j^2 = C_T$$

uniformly in n. Finally for each $\epsilon > 0$,

$$\begin{aligned}
& \lim_{n\to\infty} \sup_{|t|\le T} \left| \left[\sum_{j=1}^{n} (\phi_{n,j}(t) - 1) \right] + \frac{t^2}{2} \right| \\
\le \;& \lim_{n\to\infty} \sup_{|t|\le T} \sum_{j=1}^{n} |\phi_{n,j}(t) - 1 + \frac{\sigma_j^2 t^2}{2 s_n^2}| \\
= \;& \lim_{n\to\infty} \sup_{|t|\le T} \sum_{j=1}^{n} \left| \int \left[\exp[i\, t\, \frac{x}{s_n}] - 1 - \, i\, t\, \frac{x}{s_n} + \frac{t^2 x^2}{2 s_n^2} \right] d\alpha_j \right| \\
\le \;& \lim_{n\to\infty} \sup_{|t|\le T} \sum_{j=1}^{n} \left| \int_{|x|<\epsilon s_n} \left[\exp[i\, t\, \frac{x}{s_n}] - 1 - \, i\, t\, \frac{x}{s_n} + \frac{t^2 x^2}{2 s_n^2} \right] d\alpha_j \right| \\
& \quad + \lim_{n\to\infty} \sup_{|t|\le T} \sum_{j=1}^{n} \left| \int_{|x|\ge\epsilon s_n} \left[\exp[i\, t\, \frac{x}{s_n}] - 1 - \, i\, t\, \frac{x}{s_n} + \frac{t^2 x^2}{2 s_n^2} \right] d\alpha_j \right| \\
\le \;& \lim_{n\to\infty} C_T \sum_{j=1}^{n} \int_{|x|<\epsilon s_n} \frac{|x|^3}{s_n^3}\, d\alpha_j \\
& \quad + \lim_{n\to\infty} C_T \sum_{j=1}^{n} \int_{|x|\ge\epsilon s_n} \frac{x^2}{s_n^2}\, d\alpha_j \\
\le \;& \epsilon C_T \limsup_{n\to\infty} \sum_{j=1}^{n} \int \frac{x^2}{s_n^2}\, d\alpha_j \\
& \quad + \lim_{n\to\infty} C_T \sum_{j=1}^{n} \int_{|x|\ge\epsilon s_n} \frac{x^2}{s_n^2}\, d\alpha_j \\
= \;& \epsilon C_T
\end{aligned}$$

by Lindeberg's condition. Since $\epsilon > 0$ is arbitrary the result is proven.

Exercise 3.15. Lyapunov's condition is the following: for some $\delta > 0$

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \int |x|^{2+\delta}\, d\alpha_j = 0$$

Prove that Lyapunov's condition implies Lindeberg's condition.

Exercise 3.16. Consider the case of mutually independent random variables $\{X_j\}$, where $X_j = \pm a_j$ with probability $\frac{1}{2}$. What do Lyapunov's and Lindeberg's conditions demand of $\{a_j\}$? Can you find a sequence $\{a_j\}$ that does not satisfy Lyapunov's condition for any $\delta > 0$ but satisfies Lindeberg's condition? Try to find a sequence $\{a_j\}$ such that the central limit theorem is not valid.

Accompanying Laws.

As we stated in the previous section, we want to study the behavior of the sum of a large number of independent random variables. We have k_n independent random variables $\{X_{n,j} : 1 \le j \le k_n\}$ with respective distributions $\{\alpha_{n,j}\}$. We are interested in the distribution μ_n of $Z_n = \sum_{j=1}^{k_n} X_{n,j}$. One important assumption that we will make on the random variables $\{X_{n,j}\}$ is that no single one is significant. More precisely for every $\delta > 0$,

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} P[\,|X_{n,j}| \ge \delta\,] = \lim_{n\to\infty} \sup_{1\le j\le k_n} \alpha_{n,j}[\,|x| \ge \delta\,] = 0$$

The condition is referred to as uniform infinitesimality.

The following construction will play a major role. If α is a probability distribution on the line and $\hat{\alpha}(t)$ is its characteristic function, for any nonnegative real number $a > 0$, $\psi_a(t) = \exp[a(\phi(t) - 1)]$ is again a characteristic distribution. In fact, if we denote by α^k the k-fold convolution of α with itself, ψ_a is seen to be the characteristic function of the proabaility distribution

$$e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} \alpha^k$$

which is a convex combination α^k with weights $e^{-a}\frac{a^j}{j!}$. We use the construction mostly with $a = 1$. If we denote the probability distribution with characteristic function $\psi_a(t)$ by $e_a(\alpha)$ one checks easily that $e_{a+b}(\alpha) = e_a(\alpha) * e_b(\alpha)$. In particular $e_a(\alpha) = e_{\frac{a}{n}}(\alpha)^n$. Probability distributions β that can be written for each $n \geq 1$ as the n-fold convolution β_n^n of some probability distribution β_n are called infinitely divisible. In particular for every $a \geq 0$ and α, $e_a(\alpha)$ is an infinitely divisible probability distribution. These are called compound Poisson distributions. A special case when $\alpha = \delta_1$ the degenerate distribution at 1, we get for $e_a(\delta_1)$ the usual Poisson distribution with parameter a. We can interpret $e_a(\alpha)$ as the distribution of the sum of a random number of independent random variables withcommon distrribution α. The random n has a distribution which is Poisson with parameter a and is independent of the random variables involved in the sum.

In order to study the distribution μ_n of Z_n it will be more convenient to replace $\alpha_{n,j}$ by an infinitely divisible distribution $\beta_{n,j}$. This is done as follows. We define

$$a_{n,j} = \int_{|x|\leq 1} x \, d\alpha_{n,j},$$

$\alpha'_{n,j}$ as the translate of $\alpha_{n,j}$ by $-a_{n,j}$, i.e.

$$\alpha'_{n,j} = \alpha_{n,j} * \delta{-a_{n,j}},$$

$$\beta'_{n,j} = e_1(\alpha_{n,j}),$$

$$\beta_{n,j} = \beta'_{n,j} * a_{n,j}$$

and finally

$$\lambda_n = \prod_{j=1}^{k_n} \beta_{n,j}$$

A main tool in this subject is the following theorem. We assume always that the uniform infinitesimality condition holds.

Theorem 3.19. **(Accompanying Laws.)** In order that, for some constants A_n, the distribution $\mu_n * \delta_{A_n}$ of $Z_n + A_n$ may converge to the limit μ it is necessary and sufficient that, for the same constants A_n, the distribution $\lambda_n * \delta_{A_n}$ converges to the same limit μ.

Proof. First we note that, for any $\delta > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{1\le j\le k_n} |a_{n,j}| &= \limsup_{n\to\infty} \sup_{1\le j\le k_n} \left|\int_{|x|\le 1} x\, d\alpha_{n,j}\right| \\
&\le \limsup_{n\to\infty} \sup_{1\le j\le k_n} \left|\int_{|x|\le \delta} x\, d\alpha_{n,j}\right| \\
&\qquad + \limsup_{n\to\infty} \sup_{1\le j\le k_n} \left|\int_{\delta<|x|\le 1} x\, d\alpha_{n,j}\right| \\
&\le \delta + \limsup_{n\to\infty} \sup_{1\le j\le k_n} \alpha_{n,j}[\,|x|\ge \delta\,] \\
&= \delta
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} |a_{n,j}| = 0$$

This means that $\alpha'_{n,j}$ are uniformly infinitesimal just as $\alpha_{n,j}$ were. Let us suppose that n is so large that $\sup_{1\le j\le k_n} |a_{n,j}| \le \frac{1}{4}$. The advantage in going from $\alpha_{n,j}$ to $\alpha'_{n,j}$ is that we can calculate

$$\begin{aligned}
a'_{n,j} &= \int_{|x|\le 1} x\, d\alpha'_{n,j} \\
&= \int_{|x-a_{n,j}|\le 1} (x - a_{n,j})\; d\alpha_{n,j} \\
&= \int_{|x-a_{n,j}|\le 1} x\, d\alpha_{n,j} - a_{n,j}\,\alpha_{n,j}[\,|x-a_{n,j}|\le 1\,] \\
&= \int_{|x-a_{n,j}|\le 1} x\, d\alpha_{n,j} - a_{n,j} + \alpha_{n,j}[\,|x-a_{n,j}| > 1\,]
\end{aligned}$$

and estimate $|a'_{n,j}|$ by

$$|a'_{n,j}| \le C\alpha_{n,j}[\,|x| \ge \frac{3}{4}\,] \le C\alpha'_{n,j}[\,|x| \ge \frac{1}{2}\,]$$

In other words we may assume with out loss of generality that $\alpha_{n,j}$ satisfy the bound

$$|a_{n,j}| \le C\alpha_{n,j}[\,|x| \ge \frac{1}{2}\,]$$

and forget all about the change from $\alpha_{n,j}$ to $\alpha'_{n,j}$.

Then, just as in the proof of the Lindeberg theorem, we proceed to estimate

$$\begin{aligned}
\lim_{n\to\infty}\sup_{|t|\le T}|\log\hat\lambda_n(t) &- \log\hat\mu_n(t)| \\
&\le \lim_{n\to\infty}\sup_{|t|\le T}|\sum_{j=1}^{k_n}[\log\hat\alpha_{n,j}(t)-(\hat\alpha_{n,j}(t)-1)]| \\
&\le \lim_{n\to\infty}\sup_{|t|\le T}\sum_{j=1}^{k_n}|\log\hat\alpha_{n,j}(t)-(\hat\alpha_{n,j}(t)-1)| \\
&\le \lim_{n\to\infty}\sup_{|t|\le T}C\sum_{j=1}^{k_n}|\hat\alpha_{n,j}(t)-1|^2 \\
&= 0
\end{aligned}$$

provided we prove that if either λ_n or μ_n has a limit after translation by some constants A_n, then

$$\sup_n\sup_{|t|\le T}\sum_{j=1}^{k_n}|\hat\alpha_{n,j}(t)-1|\le C<\infty \tag{3.5}$$

Let us first suppose that λ_n has a weak limit as $n\to\infty$ after translation by A_n. The characteristic functions

$$\exp[\sum_{j=1}^{k_n}(\hat\alpha_{n,j}(t)-1))+itA_n]=\exp[f_n(t)]$$

have a limit, which is again a characteristic function. Since the limiting characteristic function is continuous and equals 1 at $t=0$, and the convergence is uniform near 0, on some small interval $|t|\le T_0$ we have the bound

$$\sup_n\sup_{|t|\le T_0}[1-\mathrm{Real}f_n(t)]\le C$$

or equivalently

$$\sup_n\sup_{|t|\le T_0}\sum_{j=1}^{k_n}\int(1-\cos t\,x)\,d\alpha_{n,j}\le C$$

and from the subadditivity property $(1-\cos 2\,t\,x\,) \leq 4(1-\cos\,t\,x)$ this bound extends to arbitrary interval $|t| \leq T$,

$$\sup_n \sup_{|t|\leq T} \sum_{j=1}^{k_n} \int (1-\cos t\,x\,)\, d\alpha_{n,j} \leq C_T$$

If we integrate the inequality with respect to t over the interval $[-T,T]$ and divide by $2T$, we get

$$\sup_n \sum_{j=1}^{k_n} \int (1-\frac{\sin T\,x}{Tx})\, d\alpha_{n,j} \leq C_T$$

from which we can conclude that

$$\sup_n \sum_{j=1}^{k_n} \alpha_{n,j}[\,|x| \geq \delta\,] \leq C_\delta < \infty$$

for every $\delta > 0$ by choosing $T = \frac{2}{\delta}$. Moreover using the inequality $(1-\cos x) \geq c\,x^2$ valid near 0 for a suitable choice of c we get the estimate

$$\sup_n \sum_{j=1}^{k_n} \int_{|x|\leq 1} x^2\, d\alpha_{n,j} \leq C < \infty$$

Now it is straight forward to estimate, for $t \in [-T,T]$,

$$\begin{aligned}
|\hat{\alpha}_{n,j}(t)-1| &= \left| \int [\exp(i\,t\,x\,)-1]\, d\alpha_{n,j} \right| \\
&= \left| \int_{|x|\leq 1} [\exp(i\,t\,x\,)-1]\, d\alpha_{n,j} \right| \\
&\quad + \left| \int_{|x|>1} [\exp(i\,t\,x\,)-1]\, d\alpha_{n,j} \right| \\
&\leq \left| \int_{|x|\leq 1} [\exp(i\,t\,x\,)-1-i\,t\,x]\, d\alpha_{n,j} \right| \\
&\quad + \left| \int_{|x|>1} [\exp(i\,t\,x\,)-1]\, d\alpha_{n,j} \right| + T\,|a_{n,j}| \\
&\leq C_1 \int_{|x|\leq 1} x^2\, d\alpha_{n,j} + C_2\, \alpha_{n,j}\,[x : |x| \geq \frac{1}{2}\,]
\end{aligned}$$

which proves the bound of equation 3.5.

Now we need to establish the same bound under the assumption that μ_n has a limit after suitable translations. For any probability measure α we define $bar\alpha$ by $\bar{\alpha}(A) = \alpha(-A)$ for all Borel sets. The distribution $\alpha * \bar{\alpha}$ is denoted by $|\alpha|^2$. The characteristic functions of $\bar{\alpha}$ and $|\alpha|^2$ are respectively $\bar{\hat{\alpha}}(t)$ and $|\hat{\alpha}(t)|^2$ where $\hat{\alpha}(t)$ is the characteristic function of α. An elementary but important fact is $|\alpha * A|^2 = |\alpha|^2$ for any translate A. If μ_n has a limit so does $|\mu_n|^2$. We conclude that the functions

$$\lim_{n\to\infty} |\hat{\mu}_n(t)|^2 = \lim_{n\to\infty} \prod_{j=1}^{k_n} |\hat{\alpha}_{n,j}(t)|^2$$

exists and defines a characteristic function which is continuous at 0 with a value of 1. Moreover because of uniform infinitesimality,

$$\lim_{n\to\infty} \inf_{|t|\le T} |\alpha_{n,j}(t)| = 1$$

It is easy to conclude that there is a $T_0 > 0$ such that, for $|t| \le T_0$,

$$\sup_n \sup_{|t|\le T_0} \sum_{j=1}^{k_n} [1 - |\hat{\alpha}_{n,j}(t)|^2] \le C_0 < \infty$$

and by subadditivity for any finite T,

$$\sup_n \sup_{|t|\le T} \sum_{j=1}^{k_n} [1 - |\hat{\alpha}_{n,j}(t)|^2] \le C_T < \infty$$

providing us with the estimates

$$\sup_n \sum_{j=1}^{k_n} |\alpha_{n,j}|^2 [\,|x| \ge \delta\,] \le C_\delta < \infty \tag{2}$$

for any $\delta > 0$, and

$$\sup_n \sum_{j=1}^{k_n} \int\int_{|x-y|\le 2} (x-y)^2 d\alpha_{n,j}(x)\, d\alpha_{n,j}(y) \le C < \infty \tag{3}$$

We now show that (2) and (3) imply (1).

$$\begin{aligned}
& |\alpha_{n,j}|^2[\,x : |x| \geq \frac{\delta}{2}\,] \\
\geq\ & \int_{|x|\leq\frac{\delta}{2}} \alpha_{n,j}\,[x : |x-y| \geq \frac{\delta}{2}\,]\, d\alpha_{n,j}(y) \\
\geq\ & \alpha_{n,j}\,[\,x : |x| \geq \delta\,]\,\alpha_{n,j}\,[\,x : |x| \leq \frac{\delta}{2}\,] \\
\geq\ & \frac{1}{2}\,\alpha_{n,j}\,[\,x : |x| \geq \delta\,]
\end{aligned}$$

by uniform infinitesimality. Therfore (2) implies

$$(2') \qquad \sup_n \sum_{j=1}^{k_n} \alpha_{n,j}\,[\,x : |x| \geq \delta\,] \leq C_\delta < \infty$$

We now turn to

$$\begin{aligned}
& \int\int_{|x-y|\leq 2} (x-y)^2 d\alpha_{n,j}(x)\, d\alpha_{n,j}(y) \\
\geq\ & \Big\{\alpha_{n,j}\,[y : |y| \leq 1\,]\Big\}\Big\{\inf_{|y|\leq 1}\int_{|x|\leq 1}(x-y)^2 d\alpha_{n,j}(x)\Big\}
\end{aligned}$$

The first term on the right can be assumed to be atleast $\frac{1}{2}$ by uniform infinitesimality. The second term

$$\begin{aligned}
\int_{|x|\leq 1}(x-y)^2 d\alpha_{n,j}(x) \ &\geq\ \int_{|x|\leq 1} x^2 d\alpha_{n,j}(x) - 2y\int_{|x|\leq 1} x\, d\alpha_{n,j}(x) \\
&\geq\ \int_{|x|\leq 1} x^2 d\alpha_{n,j}(x) - 2\Big|\int_{|x|\leq 1} x\, d\alpha_{n,j}(x)\Big| \\
&\geq\ \int_{|x|\leq 1} x^2 d\alpha_{n,j}(x) - C\alpha_{n,j}\,[x : |x| \geq \frac{1}{2}\,]
\end{aligned}$$

The last step is a consequence of what we assumed in the beginning with out loss of generality i.e.

$$\int_{|x|\leq 1} x\, d\alpha_{n,j}(x) \leq C\alpha_{n,j}\,[x : |x| \geq \frac{1}{2}\,]$$

Because of $(2')$ we can now assert

$$(3') \qquad \sup_n \sum_{j=1}^{k_n} \int_{|x|\leq 1} x^2\, d\alpha_{n,j} \leq C < \infty$$

One can now derive (3) from (2′) and (3′) as in the earlier part.

Exercise 3.17. Let $k_n = n^2$ and $\alpha_{n,j} = \delta_{\frac{1}{n}}$ for $1 \leq j \leq n^2$. $\mu_n = \delta_n$ and show that without centering $\lambda_n * \delta_{-n}$ converges to a different limit.