Advanced Probability

Fall 99

Final Examination(Take Home)

Answer all questions
Due before Jan 10 (If grades can wait till Jan 25)
Due before Dec 20(If grades are needed by Jan 5)

Q1. Using the strong law of large numbers and binary expansion construct a distribution function $F(x)$ such that

(i)$F(0) = 1$ and $F(1) = 1$

(ii) $F(x)$ is continuous.

(iii) The measure α on $[0, 1]$ corresponding to F is singular i.e there is set A such that $\alpha(A) = 1$ while $\mu(A) = 0$ where μ is the Lebesgue measure.

Q2. If X_n is the random variable that takes the value $\pm a_n$ with probability $\frac{1}{2}$ and

$$\sigma_n^2 = a_1^2 + \cdots + a_n^2$$

is the variance of $S_n = X_1 + \cdots + X_n$ we write

$$\frac{S_n}{\sigma_n} = X_{n,1} + \cdots + X_{n,n}$$

When is $X_{n,j}$ uniformly infinitesimal?
When is the Lindeberg condition satisfied?
Express your answers as conditions on the sequence a_n ?
Does $a_n = 2^n$ or $a_n = 2^{-n}$ or $a_n = n$ satisfy them?

Q3. The law of the iterated logarithm states that if $X_1, X_2, \cdots$ are independent identically distributed random variables mean 0 and variance 1

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = +1$$

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = -1$$

with probability 1. Does this mean that $\frac{1}{2}$ is necessairly a limit point? Why?

Q4. $\{\pi_{i,j}\}$ are the transition probabilities of an irreducible Markov chain on the positive integers $i \geq 1$. If there are numbers $pi > 0$ such that

$$\sum_i p_i \pi_{i,j} = p_j$$

for every j, show that the chain is positive recurrent if and only if

$$\sum_j p_j < \infty.$$

Q5. Use the criterion of Q4. to determine the values of the parameters $0 < \alpha < 1$ and β for which the following chain is positive recurrent.

$$\pi_{i,j} = \begin{cases} \alpha + \frac{\beta}{i+4} & \text{if } \ j = 1+1 \text{ and } i \ \text{ is large} \\ 1 - \pi_{i,i+1} & \text{if } \ j = i-1 \ \text{ and } i \text{ is large} \\ 0 & \text{if } \ j \neq i \pm 1 \ \text{ with } \ i \text{ large} \end{cases}$$

with $\pi_{i,j}$ defined arbitrarily for small i so that the chain is irreducible and $\pi_{i,j} = 0$ unless $|j-i| \leq 1$