3. Operators on a Hilbert Space.

A Hilbert space \( H \) is a vector space over the real or complex scalars endowed with an inner product \( \langle \, , \rangle \) than maps \( H \times H \) into \( \mathbb{R} \) or \( \mathbb{C} \) that satisfies the following properties.

1. \( \langle x, y \rangle = \langle y, x \rangle \) and \( \langle x, y \rangle \) is linear in \( x \), i.e. \( \langle a_1 x_1 + a_2 x_2, y \rangle = a_1 \langle x_1, y \rangle + a_2 \langle x_2, y \rangle \) and semilinear in \( y \), that is \( \langle x, a_1 y_1 + a_2 y_2 \rangle = a_1 \langle x, y_1 \rangle + a_2 \langle x, y_2 \rangle \)

2. \( \langle x, x \rangle \geq 0 \) and is equal to 0 if and only if \( x = 0 \). It follows that \( \| x \| = \sqrt{\langle x, x \rangle} \) is a norm and

3. \( H \) is complete under this norm, as a metric space with \( d(x, y) = \| x - y \| \).

We first note that \( \langle ax + by, ax + by \rangle = \| a \|^2 \langle x, x \rangle + \| b \|^2 \langle y, y \rangle + 2RPa\overline{b} \angle x, y \rangle \geq 0 \) for all values of \( a \) and \( b \). This forces

\[
|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle
\]

and

\[
\| x + y \| \leq \| x \| + \| y \|
\]

for all \( x, y \in H \). This makes \( d(x, y) = \| x - y \| \) in to a metric and \( H \) is assumed to be complete under this metric.

**Example 1.** \( H = L_2[0, 1] \). \( \langle f, g \rangle = \int_0^1 f(s) \overline{g(s)} ds \)

**Example 2.** \( H = l_2[\mathbb{Z}^+] \). \( \langle \{a_n\}, \{b_n\} \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \)

We say that \( x \) and \( y \) are orthogonal or \( x \perp y \) if \( \langle x, y \rangle = 0 \). A collection \( \{x_\alpha\} \) is mutually orthogonal if \( \langle x_\alpha, x_\beta \rangle = 0 \) for \( \alpha \neq \beta \). It is an orthonormal family if in addition \( \|x_\alpha\| = 1 \) for every \( \alpha \). Any two vectors in an orthonormal family are at a distance \( \sqrt{2} \).

In a separable Hilbert space any orthonormal set is either finite or countable. A maximal collection of orthonormal \( \{e_\alpha\} \) vectors in \( H \) is a basis and

\[
x = \sum_\alpha \langle x, e_\alpha \rangle e_\alpha
\]

is a convergent expansion with

\[
\| x \|^2 = \langle x, x \rangle = \sum_\alpha |\langle x, e_\alpha \rangle|^2
\]

For any subspace \( K \subset H \) there is the orthogonal complement \( K^\perp = \{ y : y \perp K \} \). \( (K^\perp)^\perp = K \). \( H = K \oplus K^\perp \). If \( \Lambda(x) \) is a bounded linear functional on \( H \) there is a unique \( y \in H \) such that \( \Lambda(x) = \langle x, y \rangle \). To prove it let us look at the null space \( K = \{ x : \Lambda(x) = 0 \} \). It has codimension 1 and has \( x_0 \) that is orthogonal to \( K \) and \( \|x_0\| = 1 \) with \( \Lambda(x_0) = c \neq 0 \). Claim \( \Lambda(x) = \langle x, cx_0 \rangle \). True on \( K \) and true for \( x = x_0 \). They span \( H \).
**Weak topology.** \( x_n \rightarrow x \) if \( \langle y, x_n \rangle \rightarrow \langle y, x \rangle \) for all \( y \in \mathcal{H} \). The unit ball \( \{ x : \| x \| \leq 1 \} \) is compact in the weak topology. That is, given any bounded sequence \( x_n \) with \( \| x_n \| \leq C \) there is a subsequence \( x_{n_j} \rightarrow x \). To see this we can assume \( \mathcal{H} \) is separable. It is enough to check it for a countable dense set of \( y \in \mathcal{H} \). But for each \( y, \langle y, x_n \rangle \) is bounded and we can extract a subsequence \( x_{n_j} \), such that \( \langle y, x_{n_j} \rangle \) has a limit. Diagonalization works. We get a subsequence that works for a countable dense set and hence for all \( y \). The limit is a bounded linear functional of \( y \) and is \( \langle y, x_0 \rangle \) for some \( x_0 \in \mathcal{H} \).

**Orthogonal Projection.** If \( K \subset \mathcal{H} \) then \( \mathcal{H} = K \oplus K^\perp \) and \( x \) can be uniquely decomposed as \( x = x_1 + x_2 \) with \( x_1 \in K \) and \( x_2 \in K^\perp \). The maps \( P_i : x \rightarrow x_i \) are self adjoint, satisfy \( P_i^2 = P_i \), \( P_1 P_2 = P_2 P_1 = 0 \) and \( P_1 + P_2 = I \). The infimum inf\( y \in K \| y - x \| \) is attained when \( y = P_1 x \).

**Problem. 1.** If \( x_n \rightarrow x \) then \( \| x \| \leq \lim \inf_{n \rightarrow \infty} \| x_n \| \). If \( x_n \rightarrow x \) and \( \| x_n \| \rightarrow \| x \| \) then \( \| x_n - x \| \rightarrow 0 \).

**Linear Operators on \( \mathcal{H} \).** A map \( T \) from one Hilbert space \( \mathcal{H} \) to another Hilbert space \( K \) is a bounded linear operator if it is linear i.e. \( T(ax + by) = aTx + bTy \) and bounded i.e. \( \| Tx \| \leq C \| x \| \). A linear map is continuous if and only if it is bounded. \( \| T \| = \sup \| Tx \| \leq 1 \) compact in \( K \). The adjoint \( T^* \) of a bounded linear operator \( T : \mathcal{H} \rightarrow \mathcal{H} \) is defined by \( \langle T^* x, y \rangle = \langle x, Ty \rangle \). One checks that \( (aT_1 + bT_2)^* = \bar{a}T_1^* + \bar{b}T_2^* \) and \( T_1^* T_2^* = T_2^* T_1^* \). An operator \( T \) is self adjoint if \( T^* = T \) i.e. \( \langle Tx, y \rangle = \langle x, Ty \rangle \). In general the product \( T_1 T_2 \) of two self adjoint operators is not self adjoint unless they commute, i.e \( T_1 T_2 = T_2 T_1 \). If \( T \) is self adjoint so is any \( p(T) \) for any polynomial \( p \) with real coefficients.

The resolvent set of an operator \( T \) in Hilbert Space over the complex numbers is \( z \in \mathbb{C} \) such that \( (zI - T)^{-1} \) exists as a bounded operator., i.e. \( (zI - T) \) is one to one, onto and ( therefore has a bounded inverse), its complement is the spectrum \( S(T) \).

If \( z \in S(T) \) then \( |z| \leq \| T \| \). If \( |z| > \| T \| \),

\[
(zI - T)^{-1} = z^{-1}(I - \frac{T}{z})^{-1} = \sum_{n \geq 0} \frac{T^n}{z^{n+1}}
\]

exists as a bounded operator and so \( z \notin S(T) \). If \( S(T) \) is empty \( (zI - T)^{-1} \) is entire and tends to 0 at \( \infty \). Therefore \( (I - \frac{T}{z})^{-1} \equiv 0 \). Cannot be!

If \( zI - T \) may not be invertible because it has a null space i.e nontrivial solutions exist for \( Tx = zx \) where \( z \) is a complex scalar. Then \( z \in S(T) \) and \( z \) is an eigenvalue with \( x \) as the eigenvector.

If \( T \) is a self-adjoint operator \( S(T) \subset [-\| T \|, \| T \|] \subset \mathbb{R} \). It is enough to show \( z = a + ib \notin S(T) \) if \( b \neq 0 \).

**Problem. 2.** Show that for any bounded operator \( T \), if \( N(T) = \{ x : Tx = 0 \} \) is the null space and \( R(T) = \{ y : y = Tx \} \) for some \( x \) is the range then \( N(T^*) = R(T) \).

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To prove $z = a + ib \notin S(T)$ it is enough to show that $Tx = zx$ has no nonzero solution and that $R(T - zI)$ is closed. Then it can not be a proper subspace because then the orthogonal complement which is the null space of $T^* - zI = T - zI$ would be nontrivial. We next need to prove that the range is dense. An inequality of the form $||TzI)x|| \geq c||x||$ is enough, because if $y_n = (T - zI)x_n$ has a limit $y$ then $x_n$ will be a Cauchy sequence with a limit $x$ and $(zI - T)x = y$.

\[(zI - T)x, (zI - T)x) = \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - \langle (a + ib)x, Tx \rangle - \langle Tx, (a + ib)x \rangle\]

\[= \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - (a + ib)\langle Tx, x \rangle - (a - ib)\langle Tx, x \rangle\]

\[= \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - 2a\langle Tx, x \rangle\]

\[\geq \|b\|^2\|x\|^2 + \|Tx - ax\|^2\]

An operator $T : \mathcal{H} \to \mathcal{K}$ is completely continuous or compact if any bounded sequence $x_n$ has a subsequence $x_{n_j}$ such that $Tx_{n_j}$ converges. In other words the image under $T$ of the unit ball $||x|| \leq 1$ in $\mathcal{H}$ is compact in $\mathcal{K}$ Often $\mathcal{K} = \mathcal{H}$.

An eigenvalue $\lambda$ of an operator $T$ from $\mathcal{H} \to \mathcal{H}$ is one for which $Tx = \lambda x$ has a nontrivial solution and the corresponding $x$ is the eigenvector.

**Theorem.** Let $A$ be a self adjoint compact operator from $\mathcal{H} \to \mathcal{H}$. Then there are eigenvalues and eigenspaces

\[E_\lambda = \{x : Ax = \lambda x\}\]

that are nontrivial only for a countable set $\{\lambda_j\} \subset \mathbb{R}$ such that for $\lambda_j \neq 0$, $E_{\lambda_j}$ are finite dimensional and the only point of accumulation of $\{\lambda_j\}$ is 0. $E_0$ itself can be trivial, or nontrivial of finite or infinite dimension. $\{E_{\lambda_j}\}$ are mutually orthogonal and

\[\mathcal{H} = \oplus E_{\lambda_j}\]

**Proof.** Let $\lambda = \sup_{||x|| \leq 1} \langle Ax, x \rangle$. Clearly $\lambda \geq 0$ and assume that $\lambda > 0$. There is a sequence $x_n$ with $||x_n|| \leq 1$ and $\langle Ax_n, x_n \rangle \to \lambda$. Choose a subsequence $x_{n_j}$ that converges weakly to $x_0$. Then $Ax_{n_j} \to Ax_0$ must converge strongly (in norm) to $Ax_0$. Implies $\langle Ax_{n_j}, x_{n_j} \rangle \to \langle Ax_0, x_0 \rangle = \lambda$. If $||x_0|| = c < 1$, $\langle Ac^{-1}x_0, c^{-1}x_0 \rangle = c^{-2}\lambda > \lambda = \sup_{||x|| \leq 1} \langle Ax, x \rangle$. A contradiction. So $||x_0|| = 1$ and the supremum is attained at $x_0$. In particular for $y \perp x_0$

\[F(\epsilon) = \frac{1}{1 + \epsilon^2} \langle Ax_0 + \epsilon y, x_0 + \epsilon y \rangle \geq \lambda = F(0)\]

It follows that $F'(0) = \langle Ax_0, y \rangle = 0$. If $Ax_0 \perp y$ whenever $x_0 \perp y$, $Ax_0 = cx_0$ and $c = \langle Ax_0, x_0 \rangle = \lambda$. We can repeat the process on $\mathcal{K} = \{y : y \perp x_0\}$ and proceed to get a sequence of eigenvalues $\lambda_n > 0$, with mutually orthogonal eigenvectors $x_n$ satisfying $||x_n|| = 1$ and $Ax_n = \lambda_n x_n$. The process may send at a finite stage are proceed without end. We note that if $||x_n|| = 1$ and $\{x_n\}$ is mutually orthogonal

\[\sum_n |\langle y, x_n \rangle|^2 \leq ||y||^2\]
and \( x_n \to 0, \|Ax_n\| \to 0 \) and \( \lambda_n \to 0 \). If \( \mathcal{K}^+ \) is the span of \( \{x_n\} \), then on \( \mathcal{K}^\perp \), \( \langle Ax, x \rangle \leq 0 \). We repeat the process with \(-A\) and recover negative eigenvalues and eigenvectors corresponding to them, the eigenvectors span \( \mathcal{K}^- \) forcing \( A = 0 \) on \([\mathcal{K}^+ \oplus \mathcal{K}^-] \perp \).

A self adjoint operator \( T \) is positive semidefinite, i.e. \( (T \geq 0) \) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \).

**Theorem** If \( T \) is a self adjoint operator and if \( p(t) \) is a polynomial with real coefficients such that \( p(t) \geq 0 \) on the interval \([-\|T\|, \|T\|]\) then \( p(T) \) is positive semi definite.

**The proof proceeds along these steps.**

If \( A \geq 0 \), there is a self adjoint operator \( B \geq 0 \) that commutes with \( A \), is in fact a limit of polynomials of \( A \) such that \( B^2 = A \). By multiplying by a constant we can assume that \( 0 \leq A \leq I \). Then since

\[
\sqrt{\lambda} = \sqrt{1 - (1 - \lambda)} = 1 - \frac{1}{2} (1 - \lambda) - \sum_{n \geq 2} \frac{1 \cdot 3 \cdot (2n - 3)}{2^n n!} (1 - \lambda)^n
\]

the series

\[
\sum_{n \geq 2} \frac{1 \cdot 3 \cdot (2n - 3)}{2^n n!}
\]

converges,

\[
B = \sqrt{A} = \sqrt{1 - (1 - A)} = 1 - \frac{1}{2} (1 - A) - \sum_{n \geq 2} \frac{1 \cdot 3 \cdot (2n - 3)}{2^n n!} (1 - A)^n
\]

is well defined, is a self adjoint operator, commutes with \( A \) is a limit in operator norm of polynomials in \( A \) and \( B^2 = A \). If \( A_1 \geq 0 \) and \( A_2 \geq 0 \) are self adjoint operators that commute, then \( A_1 A_2 \) is self-adjoint and \( A_1 A_2 \geq 0 \). \( A_i = B_i^2 \) for \( i = 1, 2 \). They all mutually commute and \( A_1 A_2 = (B_1 B_2)^2 \geq 0 \).

Let the roots of \( p(t) = 0 \) be \( \{t_j\} \). They come in different types. Complex pairs \( \{a_j \pm ib_j\} \) \( \{c_j \leq -\|T\|\}, \{d_j \geq \|T\|\} \) and roots of even multiplicity \( \theta_j \in (-\|T\|, \|T\|) \). For some \( c > 0 \)

\[
p(t) = c \Pi(t - \theta_j)^{2n_j} \Pi(t - a_j)^2 + b_j^2) \Pi(t - c_j) \Pi(d_j - t)
\]

and

\[
p(T) = c \Pi(T - \theta_j I)^{2n_j} \Pi((T - a_j I)^2 + b_j^2 I) \Pi(T - c_j I) \Pi(d_j I - T) \geq 0
\]

**Remark.** If \( f \) is a continuous function on \([-\|T\|, \|T\|]\), it is a uniform limit of polynomials \( p_n(t) \) and then \( p_n(T) \) will have a limit \( f(T) \). This defines \( f(T) \) for \( f \in C([-\|T\|, \|T\|]) \).

\[
\|f(T)\| \leq \sup_{-\|T\| \leq t \leq \|T\|} |f(t)|
\]

The linear functional \( \langle f(T)x, x \rangle \) is a nonnegative linear functional having a representation

\[
\Lambda_x(f) = \int_{[-\|T\|, \|T\|]} f(t) \mu(x, x)(dt)
\]
where \( \mu(x,x) \) is a nonnegative measure of mass \( \|x\|^2 \) supported on \([-\|T\|, \|T\|]\). We define

\[
\mu(x,y) = \frac{1}{4} \left[ \mu(x+y,x+y) - \mu(x-y,x-y) \right]
\]

in the real case and in the complex case

\[
\mu(x,y) = \frac{1}{4} \left[ \mu(x+y,x+y) - \mu(x-y,x-y) - i\mu(x+iy,x+iy) + i\mu(x-iy,x-iy) \right]
\]

Now \( \int f(t)\mu(x,y)(dt) = \langle f(T)x, y \rangle \) is defined for all bounded measurable functions \( f \). Satisfies \( (fg)(T) = f(T)g(T) \).

\[
\langle f(T)g(T)x, y \rangle = \int f(t)g(t)\mu(x,y)(dt)
\]

Pass to the limit from polynomials. Use bounded convergence theorem on the right and weak limits on the left.

**Problem 3.** Show that for any \( x \in \mathcal{H}, \mu(x,x)[(S(T))^c] = 0 \)

**Hint:** Prove it first when \( S(T) \subset \{ \lambda : |\lambda| \geq \ell \} \) for some \( \ell \) and then show that it is enough.

**Problem 4.** Identify the spectral measures \( \mu(x,x)(dt) \) for a compact self-adjoint operator \( A \).

**Projection valued measures.** If \( E \subset [-\|T\|, \|T\|] \) is a Borel set then \( \chi_E(T) \) is well defined. \( \langle \chi_E(T)x, y \rangle = \int_E \mu(x,y)(dt) \). Since \( \chi_E^2 = \chi_E, \sigma(E) = \chi_E(T) \) is a projection. \( \sigma(E) \) is a projection valued measure. It satisfies

1. For any \( E \in \mathcal{B}, \sigma(E) \) is an orthogonal projection.
2. For disjoint Borel sets \( \{E_i\}, \sigma(E_i)\sigma(E_j) = 0 \) for \( i \neq j \), and \( \sigma(\cup E_i) = \sum_i \sigma(E_i) \).

**Hilbert-Schmidt Operators.** An operator \( A \) on a separable Hilbert space \( \mathcal{H} \) is Hilbert-Schmidt if for some orthonormal basis \( \{e_j\}, \sum_{i,j} |\langle Ae_i, e_j \rangle|^2 < \infty \).

**Problem 5.** Prove that the definition is independent of the basis and that all Hilbert-Schmidt operators are compact.

**Trace Class Operators.** A positive semidefinite self adjoint operator \( A \) is of trace class if \( \sum_i \langle Ae_i, e_i \rangle \) is finite for some basis. Then it is finite on any basis and \( \text{Trace} A = \sum_i \langle Ae_i, e_i \rangle \) is well defined. \( A \) is Hilbert-Schmidt if and only if \( AA^* \) or equivalently \( AA^* \) is of trace class.

**Problem 6.** Show that if \( A \) is a compact operator, the nonzero eigenvalues of \( AA^* \) and \( A^*A \) are the same and have the same multiplicity. In particular their traces are both finite and equal or both infinite.

Consider the operator on \( L_2[0,1], \)

\[
(Tf)(x) = \int_0^1 f(y)k(x,y)dy
\]
is well defined as a bounded operator, if \( \int_0^1 \int_0^1 |k(x, y)|^2 dx dy < \infty \) and is in fact Hilbert-Schmidt. It is self adjoint if \( k(x, y) = k(y, x) \) and then the eigenvalues and eigenfunctions satisfy

\[
\sum_j \lambda_j^2 = \int_0^1 \int_0^1 |k(x, y)|^2 dx dy
\]

\[
\sum_{i,j} \lambda_j f_j(x) f_j(y) = k(x, y)
\]

in \( L_2[[0,1]^2] \). If \( k(x, y) \) is continuous and positive definite (i.e. \( \{k(x_i, x_j)\} \) is a positive semidefinite matrix for any finite collection \( \{x_i\} \)), \( T \) is positive definite operator which is trace class with trace equal to \( \int_0^1 k(x, x) dx \). The convergence in (1) is uniform.

**Problem 8.** Consider the operator

\[
(T f)(x) = \int_0^1 f(y) k(x, y) dy
\]

on \( L_2[0,1] \), where \( k(x, y) = \min(x, y) - xy \). Find all the eigenvalues and eigenfunctions. Deduce the value of the sum \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).