Chapter 9

Girsanov Formula

If α is Gaussian with mean b_1 and variance a while β has the same variance but a mean b_2 the Radon-Nikodym derivative can be explicitly calculated

$$\frac{d\beta}{d\alpha}(x) = e^{-\frac{(x-b_2)^2}{2a}+\frac{(x-b_1)^2}{2a}} = e^{\frac{(b_2-b_1)(x-b_1)}{a}-\frac{(b_2-b_1)^2}{2a}}$$

This suggests that if $P \in \mathcal{I}(a,b)$ and $Q \in \mathcal{I}(a, b+ac)$ for some bounded c, then

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp[\int_{s_0}^t c(s,x(s))dy(s) - \frac{1}{2}\int_{s_0}^t \langle a(s,x(s))c(s,x(s)), c(s,x(s))\rangle ds]$$

where

$$y(t) = x(t) - \int_{s_0}^t b(s,x(s))ds$$

Theorem 9.1. *With*

$$R(t,\omega) = \exp[\int_{s_0}^t c(s,x(s))dy(s) - \frac{1}{2}\int_{s_0}^t \langle a(s,x(s))c(s,x(s)), c(s,x(s))\rangle ds]$$

if $P \in \mathcal{I}(a,b)$ then Q with $\frac{dQ}{dP}|\mathcal{F}_t = R(t,\omega)$ is in $\mathcal{I}(a,b+ac)$ and conversely if $Q \in \mathcal{I}(a,b+ac)$ then P with $\frac{dP}{dQ}|\mathcal{F}_t = \frac{1}{R(t,\omega)}$ is in $\mathcal{I}(a,b)$.

Proof. If $P \in \mathcal{I}(a,b)$ then with

$$y(t) = x(t) - \int_{s_0}^t b(s,x(s))ds$$

$$R(t,\omega) = \exp[\int_{s_0}^t c(s,x(s)) \cdot dy(s) - \frac{1}{2}\int_{s_0}^t \langle a(s,x(s))c(s,x(s)), c(s,(x(s))\rangle ds]$$

is martingale. We can define Q by $\frac{dQ}{dP}|\mathcal{F}_t = R(t,\omega)$. We can replace c by $c(s,x)+\theta$ and will have

$$\begin{aligned}
R(t,\theta,\omega) &= \exp\Big[\int_{s_0}^t (\theta + c(s,x(s)))\cdot dy(s) \\
&\qquad - \frac{1}{2}\int_{s_0}^t \langle a(s,x(s))(\theta + c(s,x(s))), (\theta + c(s,(x(s)))\rangle ds\Big] \\
&= R(t,\omega)\exp\big[\langle \theta, y(t)-y(s_0)\rangle - \int_{s_0}^t \langle a(s,x(s))c(s,x(s)),\theta\rangle \\
&\qquad - \frac{1}{2}\int_{s_0}^t \langle a(s,(x(s))\theta,\theta\rangle\big]
\end{aligned}$$

is a martingale for all θ. It is easy to see that this equivalent to

$$\begin{aligned}
&\exp\big[\langle \theta, y(t)-y(s_0)\rangle - \int_{s_0}^t \langle \theta, a(s,x(s))c(s,x(s))\rangle ds \\
&\qquad - \frac{1}{2}\int_{s_0}^t \langle a(s,(x(s))\theta,\theta\rangle ds\big] \\
&= \exp\big[\langle \theta, x(t)-x(s_0) - \int_{s_0}^t \langle \theta, b(s,x(s)) + a(s,x(s))c(s,x(s))\rangle ds \\
&\qquad - \frac{1}{2}\int_{s_0}^t \langle a(s,(x(s))\theta,\theta\rangle ds\big]
\end{aligned}$$

being a martingale with respect to $(C[s_0,T],\mathcal{F}_t,Q)$ i.e. $Q \in \mathcal{I}(a, b+ac)$. The steps can be retraced to prove the converse. □