Chapter 7

Stochastic Differential Equations

We will fix $\sigma(s,x)$ such that $\sigma(s,x)\sigma^*(s,x) = a(s,x)$ and $b(s,x)$. Assume that σ, b are uniformly bounded and Lipschitz. We have Brownian motion on some $(\omega, \mathcal{F}_t, P)$. Given (x_0, s_0) we want solve

$$x(t) = x(s_0) + \int_{s_0}^{t} \sigma(s, x(s))d\beta(s) + \int_{0}^{t} b(s, x(s))ds \tag{7.1}$$

Theorem 7.1. *Under the regularity conditions on σ, b, for given s_0, x_0, the solution $x(t) = x(t, s_0, x_0)$ exists for all $s \geq s_0$. It is unique with in the class of progressively measurable solutions.*

Proof. We do a Picard iteration. Let $x_0(s) \equiv x_0$ for $s \geq s_0$. Define recursively

$$x_n(s) = x_0 + \int_{s_0}^{t} \sigma(s, x_{n-1}(s))d\beta(s) + \int_{s_0}^{t} b(s, x_{n-1}(s))ds$$

Since $\sigma(s,x)$ and $b(s,x)$ are bounded we can prove by induction that $x_n(\cdot)$ are well defined for $s \geq s_0$ and are progressively measurable. We denote by $Z_n(s) = x_{n+1}(s) - x_n(s)$ the difference. Then

$$Z_n(t) = \int_{s_0}^{t} [\sigma(s, x_n(s)) - \sigma(s, x_{n-1}(s)]d\beta(s) + \int_{s_0}^{t} [b(s, x_n(s)) - b(s, x_{n-1}(s))]ds$$

We will try to estimate $\Delta_n(t) = E^P[\sup_{s_0 \leq s \leq t} |Z_n(s)|^2]$ in some fixed interval $s_0 \leq t \leq T$.

$$\sup_{s_0 \leq s \leq t} |Z_n(s)| \leq \sup_{s_0 \leq s \leq t} |X_n(s)| + \sup_{s_0 \leq s \leq t} |Y_n(s)|$$

where

$$X_n(t) = \int_{s_0}^{t} [\sigma(s, x_n(s)) - \sigma(s, x_{n-1}(s)]d\beta(s)$$

and

$$Y_n(t) = \int_{s_0}^{t} [b(s, x_n(s)) - b(s, x_{n-1}(s))] ds$$

$X_n(t)$ is a martingale and by Doob's inequlaity

$$E^P[\sup_{s_0 \le s \le t} \|X_n(s)\|^2] \le 4E^P[\|X_n(t)\|^2] \le 4C^2 E^P[\int_{s_0}^{t} |x_n(s) - x_{n-1}(s)|^2 ds]$$

and

$$E^P[\sup_{s_0 \le s \le t} \|Y_n(s)\|^2] \le (T - s_0)C^2 \, E^P[\int_{s_0}^{t} |x_n(s) - x_{n-1}(s)|^2 ds]$$

Therefore there is a constant $C(T)$ such that for $s_0 \le t \le T$ and $n \ge 2$,

$$\Delta_n(t) \le C(T) \int_{s_0}^{t} \Delta_{n-1}(s) ds$$

with

$$\Delta_1(t) \le C(T)(t - s_0)$$

It follows by induction that

$$\Delta_n(t) \le \frac{(t - s_0)^n [C(T)]^n}{n!}$$

Convergence of $\sum_n [\Delta(n)]^{\frac{1}{2}}$ implies that $x_n(\cdot)$ is a Cauchy sequence in $L_2(P)$ with values in the space $C[s_0, T]$. The limit $x(t)$ exists in the sense that

$$E^P[\sup_{s_0 \le t \le T} |x_n(t) - x(t)|^2] \to 0$$

implying that that $x(t)$ is indeed an almost surely continuous, progressively measurable solution of (7.1). Uniqueness is almost the same proof. If $x(t), y(t)$ are two solutions with the same initial starting point then with $Z(t) = x(t) - y(t)$, $\Delta(t) = E[\sup_{s_0 \le s \le t} |Z(s)|^2]$ satisfies

$$\Delta(t) \le C(T) \int_{s_0}^{t} \Delta(s) ds$$

with $\Delta(t) \le C(T)(t - s_0)$. We show by induction that

$$\Delta(t) \le \frac{(t - s_0)^n [C(T)]^n}{n!}$$

Letting $n \to \infty$ we arrive at $\Delta(t) = 0$ proving uniqueness. □

Once we have uniqueness we can study the properties of the unique solution which we will now call $\xi(t)$ as a function of the starting point x. In other words

$$\xi(t,x) = x + \int_{s_0}^t \sigma(s,\xi(s,x))d\beta(s) + \int_{s_0}^t b(s,\xi(s,x))ds$$

If we let $\eta(t,x,y) = \xi(t,x) - \xi(t,y)$ then using the Lipschitz condition we can estimate

$$E[|\eta(t,x,y)|^2] = \Delta(t,x,y) \le c|x-y|^2 + c(T)\int_{s_0}^t \Delta(s,x,y)ds$$

Providing an estimate of the form

$$E[|\xi(t,x) - \xi(t,y)|^2] \le c|x-y|^2 e^{c(T)(t-s_0)}$$

Lemma 7.2. *If*

$$\xi(t) = \int_0^t c(s,\omega)d\beta(s)$$

is a stochastic integral, we know that

$$E[[\xi(t)]^2] = E[\int_0^t |c(s,\omega)|^2 ds]$$

For higher moments we have the estimate

$$E[[\xi(t)]^{2k}] \le c_k E[[\int_0^t |c(s,\omega)|^2 ds]^k]$$

where c_k depends only on k.

Proof. By Itô's formula, with the help of Doob's inequality, we can find c_k such that

$$\begin{aligned}
E[[\xi(t)]^{2k}] &= k(2k-1)E[\int_0^t |c(s,\omega)|^2|\xi(s)|^{2k-2}ds] \\
&\le k(2k-1)E[\sup_{0\le s\le t}|\xi(s)|^{2k-2}\int_0^t |c(s,\omega)|^2 ds] \\
&\le k(2k-1)[E[\sup_{0\le s\le t}|\xi(s)|^{2k}]]^{\frac{k-1}{k}} E[[\int_0^t |c(s,\omega)|^2 ds]^k]^{\frac{1}{k}} \\
&\le k(2k-1)[E[\sup_{0\le s\le t}|\xi(s)|^{2k}]]^{\frac{k-1}{k}} E[[\int_0^t |c(s,\omega)|^2 ds]^k]^{\frac{1}{k}} \\
&\le c_k^{\frac{1}{k}}[E[|\xi(t)|^{2k}]]^{\frac{k-1}{k}} E[[\int_0^t |c(s,\omega)|^2 ds]^k]^{\frac{1}{k}}
\end{aligned}$$

providing

$$E[[\xi(t)]^{2k}] \le c_k E[[\int_0^t |c(s,\omega)|^2 ds]^k]$$

We need to assume first that $c(s,\omega)$ is bounded , so that all the quantities are finite. Once we have the bound we can approximate. ☐

It is now possible to estimate

$$E[|\xi(t,x)-\xi(t,y)|^{2k}] \le c(T)e^{c(T)(t-s_0)}|x-y|^{2k}$$

as well as

$$E[|\xi(t,x)-\xi(s,x)|^{2k}] \le c(T)|t-s|^{k}$$

Together they show that $\xi(t,x)$ is almost surely jointly continuous as a function of t and x. In particular if we want to solve with $\xi(s,x)=\xi_0(\omega)$, a $\mathcal{F}_{s_0}$ measurable function the solution is given by $\xi(t,\xi_0)$. This opens up the possibility of viewing solutions of SDE (7.1) as continuous random maps of $R^d \to R^d$, i.e random flows.

Remark 7.1. The solution $x(t,s_0,x_0,\omega)$ can be expressed as $x(t,s,x(s,s_0,x_0,\omega),\omega)$ i.e the solution starting at $x(s)$ from time s. This depends on the increments of the Brownian motion $\beta(s')-\beta(s)$, $t\ge s'\ge s$. They are independent of $\mathcal{F}_s$. It is therefor clear that the conditional distribution of $x(t)$ given $\mathcal{F}_s$ depends only on $x(s)$ and is given by $p(s,x(s),t,A)$, where $p(s,x,t,A)=P[x(t,s,x)\in A]$, proving that $x(t)$ is a Markov process with transition probabilities $\{p(s,x,t,A)\}$.

Remark 7.2. The independence of $\beta(t+\tau)-\beta(\tau)$ and the $\sigma-$field $\mathcal{F}_\tau$ is valid for stopping times τ just as it is for constant times s. This establishes the strong Markov property for the process $x(t)$.

For each x_0 the approximations $x_n(t)$ in the Picard iteration scheme are measurable function of the Brownian increments $\beta(t)-\beta(s_0)$ and so the limit $x(t)$ really defines a map of the Brownian path $\beta(t)-\beta(s_0)$ to the diffusion paths $x(t)$. Even if $\mathcal{F}_t$ is larger it does not play any role.

One can formulate a general uniqueness question. If $x^i(t,\omega)$, $i=1,2$ are both almost surely continuous, progressively measurable solutions to

$$x^i(t)=x_0+\int_{s_0}^t \sigma(s,x^i(s))d\beta(s)+\int_{s_0}^t b(s,x^i(s))ds$$

on some $(\Omega,\mathcal{F}_t,P,\beta(t))$ does it follow that $x^1(t)\equiv x^2(t)$ with probability 1? This is of course a property of σ and b. If the answer is yes we will say that pathwise or strong uniqueness holds for σ,b. If σ,b are Lipschitz the answer as we saw, is yes.

One can also ask the following question. Let $\{a_{i,j}(t,x)\},\{b_j(t,x)\}$ be given. If P^i for $i=1,2$ both satisfy $P^i\in\mathcal{I}(a,b)$ as well as $P^i[x(s_0)=x_0]=1$, does it follow that $P^1=P^2$. If the answer is yes we say that distribution uniqueness holds for $[a,b]$. The following theorem provides a connection.

Theorem 7.3. *If pathwise uniqueness holds for some $[\sigma,b]$ with $\sigma\sigma^*=a$, then distribution uniqueness holds for $[a,b]$.*

Remark 7.3. σ need not be a square matrix. We can use more (or less) number of Brownian motions than d dimension of $x(t)$. If σ is $d\times k$, then the rank of a can not exceed $\min(d,k)$.

Proof. 1. We start with the measure $P^i \in \mathcal{I}(a,b)$. We can assume that P^i is defined on $C[[s_0,T];R^d]$, with the natural $\mathcal{F}_t$ and coordinate function $x(\cdot)$. We can construct a measure Q^i on $C[[s_0,T];R^d \times R^n]$ with components $x(t),\beta(t)$, with marginals P^i and P the Wiener measure such that for $i=1,2$

$$x(t) = x_0 + \int_{s_0}^t \sigma(s,x(s))d\beta(s) + \int_{s_0}^t b(s,x(s))ds$$

a.e. P^i.

2. We couple the two processes by constructing a Q on $C[[s_0,T];R^d \times R^d \times R^n$ with components $x^1(t),x^2(t),\beta(t)$ such that $[x^i(\cdot),\beta(\cdot)]$ are distributed according to Q^i. Then

$$x^i(t) = x_0 + \int_{s_0}^t \sigma(s,x^i(s))d\beta(s) + \int_{s_0}^t b(s,x^i(s))ds$$

3. If pathwise uniqueness holds $x^1(t) \equiv x^2(t)$ implying $P^1 = P^2$.

4. We now turn to the construction of the coupling. Let us define $\pi^i[dx|\beta]$ as the conditional distribution of $x(\cdot)$ given β under Q^i. Define

$$Q = \pi^1(dx^1|\beta) \otimes \pi^2(dx^2|\beta) \otimes P(d\beta)$$

$\square$