Chapter 5

Stochastic Integrals and Itô's formula.

We will call an Itô process a progressively measurable almost surely continuous process $x(t,\omega)$ with values in R^d, defined on some $(\Omega, \mathcal{F}_t, P)$ that is related to progressively measurable bounded functions $[a(s,\omega), b(s,\omega)]$ in the following manner.

$$\exp[\langle \theta, x(t,\omega) - x(0,\omega) - \int_0^t b(s,\omega)ds\rangle - \frac{1}{2}\int_0^t \langle \theta, a(s,\omega)\theta\rangle ds]$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for all $\theta \in R^d$. A canonical example is Brownian motion that corresponds to $b(s,\omega) \equiv 0$ and $a(s,\omega) \equiv 1$ or $a(s,\omega) \equiv I$ in higher dimensions.We will abbreviate it by $x(\cdot) \in \mathcal{I}(a,b)$. Such processes are not of bounded variation unless $a \equiv 0$. In fact they have nontrivial quadratic variation.

Lemma 5.1. *If $x(\cdot)$ is a one dimensional process and $x(\cdot) \in \mathcal{I}(a,b)$ then*

$$\lim_{n\to\infty} \sum_{j=1}^{n} |x(\frac{jT}{n}) - x(\frac{(j-1)T}{n})|^2 = \int_0^T a(s,\omega)ds$$

in probability and in $L_1(P)$.

Proof. If $y(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds$, then $y(\cdot) \in \mathcal{I}(a,0)$ and the difference between $x(\cdot)$ and $y(\cdot)$ is a continuous function of bounded variation. it is therefore sufficient to show that

$$\lim_{n\to\infty} \sum_{j=1}^{n} |y(\frac{jT}{n}) - y(\frac{(j-1)T}{n})|^2 = \int_0^T a(s,\omega)ds$$

If we denote by

$$Z_j = |y(\frac{jT}{n}) - y(\frac{(j-1)T}{n})|^2 - \int_{\frac{(j-1)T}{n}}^{\frac{jT}{n}} a(s,\omega)ds$$

then $E[Z_j]=0$ and for $i\neq j$, $E[Z_iZ_j]=0$. It is therefore sufficient to show

$$E[|Z_j|^2]\leq \frac{C(T)}{n^2}.$$

This follows easily from the Gaussian bound

$$E[e^{\lambda(y(t_2)-y(t_1))}]\leq e^{\frac{C\lambda^2(t_2-t_1)}{2}}$$

provided $a(s,\omega)\leq C$. We see that $E[(y(t_2)-y(t_1))^4]\leq C(t_2-t_1)^2$. ∎

This means that integrals of the form $\int_0^t e(s,\omega)dx(s,\omega)$ have to be carefully defined. Since the difference between $x(\cdot)$ and $y(\cdot)$ is of bounded variation it suffices to concentrate on $\int_0^t e(s,\omega)dx(s,\omega)$. We develop these integrals in several steps, each one formulated as a lemme.

Lemma 5.2. *Let $\mathcal{S}$ be the space of functions $e(s,\omega)$ that are uniformly bounded piecewise constant progressively measurable functions of s. In other words there are intervals $[t_{j-1},t_j)$ in which $e(s,\omega)$ is equal to $e(t_{j-1},\omega)$ which is $\mathcal{F}_{t_{j-1}}$ measurable. We define for $t_{k-1}\leq t\leq t_k$*

$$\xi(t)=\int_0^t e(s,\omega)dy(s)=\sum_{j=1}^{k-1}e(t_{j-1},\omega)[y(t_j)-y(t_{j-1})]+e(t_{k-1},\omega)[y(t)-y(t_{k-1})]$$

The following facts are easy to check.

1. *$\xi(t)$ is almost surely continuous, progressively measurable. Moreover $\xi(\cdot)\in\mathcal{I}(e^2(s,\omega)a(s,\omega),0)$.*
2. *The space $\mathcal{S}$ is linear and the map $e\to\xi$ is a linear map.*
3. $$E[\sup_{0\leq s\leq t}|\xi(s,\omega)|^2]\leq 4E[\int_0^t |e(s,\omega)|^2 a(s,\omega)ds]$$
4. *In particular if $e_1,e_2\in\mathcal{S}$, and for $i=1,2$*

   $$\xi_i(t)=\int_0^t e_i(s,\omega)dy(s)$$

   then

   $$E[\sup_{0\leq s\leq t}|\xi_1(s,\omega)-\xi_2(s,\omega)|^2]\leq 4E[\int_0^t |e_1(s,\omega)-e_2(s,\omega)|^2 a(s,\omega)ds]$$

Proof. It is easy to see that, because for $\lambda\in R$,

$$E[\exp[\lambda[y(t)-y(s)]-\frac{\lambda^2}{2}\int_s^t a(u,\omega)du]|\mathcal{F}_s]=1$$

it follows that if λ is replaced by $\lambda(\omega)$ that is bounded and $\mathcal{F}_s$ measurable then

$$E[\exp[\lambda(s,\omega)[y(t)-y(s)] - \frac{\lambda(s,\omega)^2}{2}\int_s^t a(u,\omega)du]|\mathcal{F}_s] = 1$$

We can take $\lambda(s,\omega) = \lambda e^(s,\omega)$. This proves 1. 2 is obvious and 3 is just Doob's inequality. 4 is a restatement of 3 for the difference. □

Lemma 5.3. *Given a bounded progressively measurable function $e(s,\omega)$ it can be approximated by a sequence $e_n \in \mathcal{S}$, such that $\{e_n\}$ are uniformly bounded and*

$$\lim_{n\to\infty} E[\int_0^T |e_n(s,\omega) - e(s,\omega)|^2 ds] = 0$$

As a consequence the sequence $\xi_n(t) = \int_0^t e_n(s,\omega)dy(s)$ has a limit $\xi(t,\omega)$ in the sense

$$\lim_{n\to\infty} E[\sup_{0\le s\le t} |\xi_n(s) - \xi(s)|^2] = 0$$

It follows that $\xi(t,\omega)$ is almost surely continuous and $\xi(\cdot) \in \mathcal{I}(e^2(s,\omega)a(s,\omega))$.

Proof. It is enough to prove the approximation property. Since

$$Y_\lambda(t) = \exp[\lambda\xi_n(t) - \frac{\lambda^2}{2}\int_0^s e_n^2(s,\omega)a(s,\omega)ds]$$

are martingales and $e_n^2 a$ has uniform bound C, it follows that

$$\sup_{0\le t\le T}\sup_n E[\exp[\lambda\xi_n(t)]] \le \exp[\frac{C\lambda^2 T}{2}]$$

providing uniform integrability. We note that

$$\lim_{n,m\to\infty} E[\sup_{0\le s\le t} |\xi_n(s) - \xi_m(s)|^2] = 0$$

Now it is easy to show that $\xi_n(\cdot)$ has a uniform limit in probability and pass to the limit. To prove the approximation property we approximate first $e(s,\omega)$ by

$$e_h(s,\omega) = \frac{1}{h}\int_{(s-h)\vee 0}^s e(u,\omega)du$$

It is a standard result in real variables that $\|e_h(\cdot) - e_h(\cdot)\|_2 \to 0$ as $h \to 0$ and e_h is contunuos in s. Note that we only look back and not ahead, thus preserving progressive measurability. We can now approximate $e_h(s,\omega)$ by $e_h(\frac{[ns]}{n},\omega)$ that are again progressively measurable, but simple as well, so they are in $\mathcal{S}$. □

Lemma 5.4. *If $e(s,\omega)$ is progressively measurable and satisfies*

$$E[\int_0^T e^2(s,\omega)a(s,\omega)ds] < \infty$$

we can define on $[0,T]$,

$$\xi(t) = \int_0^t e(s,\omega)dy(s)$$

as a square integrable martingale and

$$\xi(t)^2 - \int_0^t e^2(s,\omega)a(s,\omega)ds$$

will be a martingale.

Proof. The proof is elementary. Just approximate e by truncated functions

$$e_n(s,\omega) = e(s,\omega)\mathbf{1}_{\{|e(s,\omega)|\le n\}}(\omega)$$

and pass to the limit. Again

$$\lim_{n,m\to\infty} E[\sup_{0\le s\le t} |\xi_n(s) - \xi_m(s)|^2] = 0$$

□

Remark 5.1. If $x(\cdot) \in \mathcal{I}(a,b)$ we can let $y(t) = x(t) - \int_0^t b(s,\omega)ds$ and define

$$\xi(t) = \int_0^t e(s,\omega)dx(s) = \int_0^t e(s,\omega)dy(s) + \int_0^t e(s,\omega)b(s,\omega)ds$$

If

$$E[\int_0^t b^2(s,\omega)e^2(s,\omega)ds] < \infty$$

then we can check ξ is well defined. In fact we can define for bounded progressively measurable e, c,

$$\xi(t) = \int e(s,\omega)dx(s) + \int c(s,\omega)ds$$

It is easy to check that

$$\xi(\cdot) \in \mathcal{I}(e^2(s,\omega)a(s,\omega), e(s,\omega)b(s,\omega) + c(s,\omega))$$

Recall that if $X \simeq N[\mu,\sigma^2]$ and $Y = aX + b$ then $Y \simeq N[a\mu + b, a^2\sigma^2]$.

Remark 5.2. We can have $x(t) \in R^d$ and $x(\cdot) \in \mathcal{I}(a,b)$, where $a = a(s,\omega)$ is a symmetric positive semidefinite matrix valued bounded progressively measurable function and $b = b(s,\omega)$ is an R^d valued, bounded and progressively measurable. We can the define

$$\xi(t) = \int_0^t e(s,\omega)\cdot dx(s) + \int c(s,\omega)ds$$

where $e(s,\omega)$ is a progressively measurable bounded $k \times d$ matrix and c is R^k valued, bounded and progressively measurable. The integral is defined by each component. For $1 \le i \le k$,

$$\xi_i(t) = \sum_j \int_0^t e_{i,j}(s,\omega) \cdot dx_j(s) + \int c_i(s,\omega) ds$$

The one verifies easily that

$$\xi(\cdot) \in \mathcal{I}(eae^*, eb + c)$$

Theorem 5.5. Itô's formula. *Consider a smooth function $f(t,x)$ on $[0,T] \times R^d$. Let $x(t)$ with values in R^d belong to $\mathcal{I}(a,b)$. Then almost surely*

$$\begin{aligned} f(t,x(t)) = f(0,x(0)) + \int_0^t f_s(s,x(s))ds + \int_0^t (\nabla_x f)(s,x(s)) \cdot dx(s) \\ + \frac{1}{2}\int_0^t \sum a_{i,j}(s,\omega)(D_{x_i,x_j} f)(s,x(s))ds \end{aligned}$$

Proof. Consider the $d+1$ dimensional process $Z(t) = (f(t,x(t)), x(t))$. If $\sigma \in R$ and $\lambda \in d$, then if we consider $g(t,x) = \sigma f(t,x) + \langle \lambda, x \rangle$ we know that

$$\exp[g(t,x(t)) - g(0,x(0)) - \int_0^t e^{-g}[\partial_s e^g + L_{s,\omega} e^g](s,x(s))ds]$$

is a martingale. A computation yields

$$\begin{aligned} e^{-g}[\partial_s e^g + L_{s,\omega} e^g] &= \partial_s g + L_{s,\omega} g + \frac{1}{2}\langle \nabla g, a \nabla g \rangle \\ &= \sigma \partial_s f + \sigma L_{s,\omega} f + \langle \lambda, b(s,\omega) \rangle \\ &\quad + \frac{1}{2}\langle (\sigma \nabla f + \lambda), a(s,\omega)(\sigma \nabla f + \lambda) \rangle \end{aligned}$$

Implies that $Z(t) \in \mathcal{I}(\tilde{a}, \tilde{b})$, where

$$\tilde{a} = \begin{pmatrix} \langle \nabla f, a \nabla f \rangle & (a \nabla f)^{tr} \\ (a \nabla f) & a \end{pmatrix}$$

and

$$\tilde{b} = (\partial_s f + L_{s,\omega} f, b)$$

Now we can compute that $w(\cdot) \in \mathcal{I}(A,B)$ where

$$\begin{aligned} w(t) = \int_0^t 1 \cdot df(s,x(s)) - \int_0^t (\partial_s f)(s,x(s))ds - \int_0^t (\nabla_s f)(s,x(s)) \cdot dx(s) \\ - \frac{1}{2}\int_0^t \sum_{i.j} a_{i,j}(s,\omega)(D_{x_i x_j} f)(s,x(s))ds \end{aligned}$$

If we can calculate and show that $A = 0$ and $B = 0$, this would imply that $w(t) \equiv 0$ and that proves the theorem.

$$A = (1, -\nabla f) \begin{pmatrix} \langle \nabla f, a\nabla f\rangle & (a\nabla f)^{tr} \\ (a\nabla f) & a \end{pmatrix} \begin{pmatrix} 1 \\ -\nabla f \end{pmatrix} = 0$$

$$B = \partial_s f + L_{s,\omega} f - b\cdot\nabla f - \partial_s f - \frac{1}{2}\sum_{i.j} a_{i,j}(s,\omega)(D_{x_i x_j} f) = 0$$

□

Remark 5.3. If $x(\cdot) \in \mathcal{I}(a,b)$ and $y(t) = \int_0^t \sigma(s,\omega)\cdot dx(s) + \int_0^t c(s,\omega)ds$ we saw that

$$y(\cdot) \in \mathcal{I}(\tilde{a}, \tilde{b})$$

where

$$\tilde{a} = \sigma a \sigma^*, \tilde{b} = \sigma b + c$$

This is like linear change of variables of a Gaussian vector. $dx \simeq N[a\,dt, b\,dt]$ and $\sigma dx + c \simeq N[\sigma a\sigma^*\,dt, (\sigma b + c)\,dt]$. We can develop stochastic integrals of $y(\cdot)$ and if $dz = \sigma' dy + c' dt$ then $dz = \sigma'\,[\sigma dx + cdt] + c' dt = \sigma'\sigma dx + (\sigma' c + c')dt$. If σ is a invertible then $dy = \sigma dx + cdt$ can be inverted as $dx = \sigma^{-1}dy - \sigma^{-1}cdt$. Finally one can remember Itô's formula by the rules

$$df(t,x(t)) = f_t dt + \sum_i f_{x_i} dx_i + \frac{1}{2}\sum_{i,j} f_{x_i,x_j} dx_i dx_j$$

If $x(\cdot) \in \mathcal{I}(a,b)$ then $dx_i dx_j = a_{i,j}dt$. $(dt)^2 = dt dx_i = 0$. Because the typical paths have half a derivative (more or less) $dx \simeq \sqrt{dt}$. $dx_i dx_j$ is of the order of dt and $dxdt, (dt)^2$ are negligible.