2.1 Continuous Parameter Martingales.

$(\Omega, \mathcal{B}, P)$ is a probability space and for $t \in [0,T]$, $\mathcal{B}_t \subset \mathcal{B}$ is an increasing family of sub-σ fields, referred to as "filtration". A martingale with respect to $(\Omega, \mathcal{B}_t, P)$ is a family $\xi(t,\omega)$ with the following properties.

- For almost all ω, $\xi(t)$ is a right continuous function of t.
- For each t, $\xi(t,\omega)$ is $\mathcal{B}_t$ measurable. With right continuity it follows that $\xi(\cdot,\omega)$ is "progressively measurable" i.e for each $t > 0$, the function $\xi(s,\omega)$ as a map of $[0,t] \times \Omega \to R$ is measurable with respect to $\mathcal{B}[0,t] \times \mathcal{B}_t$ where $\mathcal{B}[0,t]$ is the Borel $\sigma-$field of $[0,t]$.
- $\xi(t,\omega) \in L_1(P)$ and for $t > s \geq 0$, $E[\xi(t)|\mathcal{B}_s] = \xi(s,\omega)$ a.e.

Remark 2.1. According to a theorem of Doob, a continuous parameter martingale, almost surely, has limits from the left and right at every t. To demand that it be right continuous, i.e. to define the value at t as the limit from the right is a matter of normalization.

Remark 2.2. By restricting the martingale $\xi(t,\omega)$ to a discrete subset $\{nh\}$ we will get a discrete parameter martingale. The usual estimates valid for martingales are valid for them, uniformly in h. We can then let $h \to 0$ and deduce analogous results for continuous parameter martingales. For example the following theorems are easily established in this manner.

Theorem 2.1. *Let $\xi(t)$ be a continuous parameter martingale on $[0,T]$. Then*

$$P[\sup_{t\in[0,T]} |\xi(t)| \geq \ell] \leq \frac{1}{\ell}\int_{[\sup_{t\in[0,T]}|\xi(t)|\geq \ell]} |\xi(T)|dP \leq \frac{1}{\ell}E[|\xi(T)|]$$

Moreover for $p > 1$,

$$\|\sup_{t\in[0,T]} |\xi(t)|\|_p \leq \frac{p}{p-1}\|\xi(T)\|_p$$

2.2 Stopping Times.

Given a filtration $\{\mathcal{B}_t\}$ we can define a stopping time relative to the filtration. A function $\tau : \Omega \to [0,\infty]$ is called a stopping time if for every $t \geq 0$ the set $\{\omega : \tau(\omega) \leq t\}$ is $\mathcal{B}_t$ measurable. Typical examples of stopping times are the first time some thing happens, like the exit time from an open set. Given a stopping time τ there is a natural sub σ-field $\mathcal{B}_\tau$ associated with it, defined by

$$A \in \mathcal{B}_\tau \;\Leftrightarrow\; A \cap \{\tau \leq t\} \in \mathcal{B}_t \quad \forall t$$

It is easy to check that any stopping time τ is measurable with respect to $\mathcal{B}_\tau$. For any t, $\tau \wedge t$ is a stopping time as well, and $\mathcal{B}_{\tau\wedge t}$ is a new filtration. If $\xi(t)$ is a martingale with respect to $\{\mathcal{B}_t\}$ so is $\xi(\tau \wedge t, \omega)$ with respect to $\mathcal{B}_{\tau\wedge t}$.

Doob's optional stopping theorem for martingales extends to the continuous case.

Theorem 2.2. *If $0 \leq \tau_1 \leq \tau_2 \leq C$ are two bounded stopping times, and $\xi(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$ then almost surely*

$$E[\xi(\tau_2)|\mathcal{B}_{\tau_1}] = \xi(\tau_1)$$

This is proved by approximating the stopping times $\tau_i, i = 1, 2$ by $\tau_i^n = \frac{[n\tau_i]+1}{n}$. Then the optional stopping theorem can be applied to the discrete martingale $\xi(\frac{j}{n})$, conclude that $E[\xi(\tau_2^n)|\mathcal{B}_{\tau_1^n}] = \xi(\tau_1^n)$ and let $n \to \infty$ to obtain our theorem.

2.3 Strong Markov Property.

Brownian motion is a process with independent increments. It is therefore, in particular, a Markov Process. That is to say, given the past history $\mathcal{B}_s$, [the σ-field generates by $\{x(u) : 0 \leq u \leq s\}$], the conditional distribution of $x(t) = x(s) + [x(t) - x(s)]$ for $t > s$ is the normal distribution with mean $x(s)$ and variance $t - s$. Since this only depends on $x(s)$ the Markov property holds. The strong Markov property extends this from constant times s to stopping times.

We begin with $(\Omega, \mathcal{B}_t, x(t), P)$, where $\mathcal{B}_t$ is an increasing family of sub-σfields, satisfying

- For each t, $x(t, \omega)$ is $\mathcal{B}_t$ measurable
- $x(t)$ is almost surely a continuous function of t.
- For $t > s >$ almost surely

$$P[x(t) \in A|\mathcal{B}_s] = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x(s))^2}{2(t-s)}} dy$$

We will call such an $x(t)$ a Brownian motion adapted to $\{\mathcal{B}_t\}$. Note that $\mathcal{B}_t$ can be larger than $\sigma\{x(s) : 0 \leq s \leq t\}$.

Theorem 2.3. *The strong Markov property holds for Brownian Motion. That is, given any stopping time τ that is almost surely finite,*

$$P[x(t+\tau) \in A|\mathcal{B}_\tau] = \int_A \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x(\tau))^2}{2t}} dy$$

Equivalently the process $x(t+\tau) - x(\tau)$ is another Brownian Motion adapted to $\mathcal{B}_{\tau+t}$, and is independent of $\mathcal{B}_\tau$.

Proof. It is enough to show that if $A \in \mathcal{B}_\tau$ and f is a bounded continuous function, , then

$$\int_A f(x(t+\tau))dP = \int_A g(x(\tau))dP$$

where

$$g(x) = \int f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

We will check it for τ that takes only a countable set of values. $\tau = t_j$ for some j. The set $A_j = A \cap \{\tau = t_j\} \in \mathcal{B}_{t_j}$. Therefore from the Markov property

$$\begin{aligned}\int_A f(x(t+\tau))dP &= \sum_j \int_{A_j} f(x(t+\tau))dP = \sum_j \int_{A_j} f(x(t+t_j))dP \\ &= \sum_j \int_{A_j} g(x(t_j))dP = \int_A g(x(\tau))dP\end{aligned}$$

If we now approximate τ by $\tau_n = \frac{[n\tau]+1}{n}$ and pass to the limit we are done. Note that here we approximate τ by $\tau_n \geq \tau$, so that $A \in \mathcal{B}_\tau \subset \mathcal{B}_{\tau_n}$. We have also used the fact that g is continuous. □

Remark 2.3. Any Markov process that has almost surely right continuous paths and $E[f(x(t)|x(s)] = g(s,t,x(s))$ where $g(s,t,x)$ is continuous in x for each fixed $s < t$, has the strong Markov property by the same argument.

2.4 Reflection Principle.

If $x(t)$ is Brownian motion

$$P[\sup_{0 \leq s \leq t} x(s) \geq \ell] = 2P[x(t) \geq \ell]$$

Let τ be the stopping time $\tau = \inf\{s : x(s) \geq \ell\}$. We are interested in $P[\tau \leq t]$. Note that $x(\tau) = \ell$. Therefore

$$\begin{aligned}P[x(t) \geq \ell] = P[x(t) \geq \ell \ \&\ \tau \leq t] &= P[\tau \leq t] \int_\ell^\infty \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(y-\ell)^2}{2(t-\tau)}} dy \\ &= \frac{1}{2} P[\tau \leq t]\end{aligned}$$

2.5 Brownian Motion as a Martingale

P is the Wiener measure on $(\Omega, \mathcal{B})$ where $\Omega = C[0,T]$ and $\mathcal{B}$ is the Borel σ-field on Ω. In addition we denote by $\mathcal{B}_t$ the σ-field generated by $x(s)$ for $0 \leq s \leq t$. It is easy to see that $x(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$, i.e. for each $t > s$ in $[0,T]$

$$E^P[x(t)|\mathcal{B}_s] = x(s) \quad \text{a.e.} \quad P \tag{2.1}$$

and so is $x(t)^2 - t$. In other words

$$E^P[x(t)^2 - t\,|\mathcal{F}_s] = x(s)^2 - s \quad \text{a.e.} \quad P \tag{2.2}$$

The proof is rather straight forward. We write $x(t) = x(s) + Z$ where $Z = x(t) - x(s)$ is a random variable independent of the past history $\mathcal{B}_s$ and is distributed as a Gaussian random variable with mean 0 and variance $t - s$. Therefore $E^P[Z|\mathcal{B}_s] = 0$ and $E^P[Z^2|\mathcal{B}_s] = t - s$ a.e P. Conversely,

Theorem 2.4. Lévy's theorem. *If P is a measure on $(C[0,T], \mathcal{B})$ such that $P[x(0) = 0] = 1$ and the the functions $x(t)$ and $x^2(t) - t$ are martingales with respect to $(C[0,T], \mathcal{B}_t, P)$ then P is the Wiener measure.*

Proof. The proof is based on the observation that a Gaussian distribution is determined by two moments. But that the distribution is Gaussian is a consequence of the fact that the paths are almost surely continuous and not part of our assumptions. The actual proof is carried out by establishing that for each real number λ

$$X_\lambda(t) = \exp\big[\lambda x(t) - \frac{\lambda^2}{2} t\big] \tag{2.3}$$

is a martingale with respect to $(C[0,T], \mathcal{B}_t, P)$. Once this is established it is elementary to compute

$$E^P\big[\exp\big[\lambda(x(t) - x(s))\big]|\mathcal{B}_s\big] = \exp\big[\frac{\lambda^2}{2}(t-s)\big]$$

which shows that we have a Gaussian Process with independent increments with two matching moments. The proof of (2.3) is more or less the same as proving the central limit theorem. In order to prove that $X_\lambda(t)$ is a martingale, we can assume with out loss of generality that $s = 0$ and show that

$$E^P\big[\exp\big[\lambda x(t) - \frac{\lambda^2}{2} t\big]\big] = 1 \tag{2.4}$$

To this end let us define successively $\tau_{0,\epsilon} = 0$,

$$\tau_{k+1,\epsilon} = \min\big[\inf\big\{s : s \ge \tau_{k,\epsilon}, |x(s) - x(\tau_{k,\epsilon})| \ge \epsilon\big\}, t\,, \tau_{k,\epsilon} + \epsilon\big]$$

Then each $\tau_{k,\epsilon}$ is a stopping time and eventually $\tau_{k,\epsilon} = t$ by continuity of paths. The continuity of paths also guarantees that $|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})| \le \epsilon$. We write

$$x(t) = \sum_{k \ge 0} [x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})]$$

and

$$t = \sum_{k \ge 0} [\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]$$

To establish (2.4) we calculate the quantity on the left hand side as

$$\lim_{n \to \infty} E^P\big[\exp\big[\sum_{0 \le k \le n} \big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{\lambda^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big]$$

and show that it is equal to 1. Let us consider the σ-field $\mathcal{F}_k = \mathcal{B}_{\tau_{k,\epsilon}}$ and the quantity

$$q_k(\omega) = E^P\big[\exp\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - [\frac{\lambda^2}{2} + C(\lambda)\epsilon][\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big|\mathcal{F}_k\big]$$

with the choice of the constant $C(\lambda)$ to be chosen later. Clearly, if we use Taylor expansion and the fact that $x(t)$ as well as $x(t)^2 - t$ are martingales

$$\begin{aligned} q_k(\omega) &\le E^P\big[1 + c(\lambda)\big[|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^3 + |\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|^2\big] - C(\lambda)\epsilon[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\Big|\mathcal{F}_k\big] \\ &\le E^P\big[1 + c(\lambda)\epsilon\,\big[|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^2 + |\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|\big] - C(\lambda)\epsilon[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big|\mathcal{F}_k\big] \\ &\le 1 \end{aligned}$$

for some suitably chosen constant $C(\lambda)$ depending on λ. By Fatou's lemma

$$E^P\big[\exp\big[\lambda x(t) - [\frac{\lambda^2}{2} + C(\lambda)\epsilon]t\big]\big] \le 1$$

Since $\epsilon > 0$ is arbitrary we prove one half of (2.4). A similar estimate will yield

$$E^P\big[\exp\big[\lambda x(t) - [\frac{\lambda^2}{2} - C(\lambda)\epsilon]t\big]\big] \ge 1$$

which can be used to prove the other half provided we show the uniform integrability of $\{\exp[\lambda x(\tau_n)]\}$. This follows from the upper bound established above. This completes the proof of the theorem. □

Remark 2.4. Theorem 2.4 fails for the process $x(t) = N(t) - t$ where $N(t)$ is the standard Poisson Process with rate 1.

Remark 2.5. One can use the Martingale inequality in order to estimate the probability $P\{\sup_{0\le s\le t}|x(s)| \ge \ell\}$. For $\lambda > 0$, by Doob's inequality

$$P\big[\sup_{0\le s\le t}\exp\big[\lambda x(s) - \frac{\lambda^2}{2}s\big] \ge A\big] \le \frac{1}{A}$$

and

$$\begin{aligned} P\big[\sup_{0\le s\le t} x(s) \ge \ell\big] &\le P\big[\sup_{0\le s\le t}[x(s) - \frac{\lambda s}{2}] \ge \ell - \frac{\lambda t}{2}\big] \\ &= P\big[\sup_{0\le s\le t}[\lambda x(s) - \frac{\lambda^2 s}{2}] \ge \lambda\ell - \lambda^2 t2\big] \\ &\le \exp[-\lambda\ell + \frac{\lambda^2 t}{2}] \end{aligned}$$

Optimizing over $\lambda > 0$, we obtain

$$P\big[\sup_{0\le s\le t} x(s) \ge \ell\big] \le \exp[-\frac{\ell^2}{2t}]$$

and by symmetry

$$P\big[\sup_{0\le s\le t}|x(s)|\ge \ell\big]\le 2\exp[-\frac{\ell^2}{2t}]$$

The estimate is not too bad because by reflection principle

$$P\big[\sup_{0\le s\le t}x(s)\ge \ell\big]=2\,P\big[x(t)\ge \ell\big]=\sqrt{\frac{2}{\pi\,t}}\int_\ell^\infty \exp[-\frac{x^2}{2\,t}]\,dx$$

Exercise 2.1. One can use the estimate above to prove the result of Paul Lévy

$$P\Big[\limsup_{\delta\to 0}\frac{\sup_{\substack{0\le s,t\le 1\\ |s-t|\le\delta}}|x(s)-x(t)|}{\sqrt{\delta\log\frac{1}{\delta}}}=\sqrt{2}\Big]=1$$

We had an exercise in the previous section that established the lower bound. Let us concentrate on the upper bound. If we define

$$\Delta_\delta(\omega)=\sup_{\substack{0\le s,t\le 1\\ |s-t|\le\delta}}|x(s)-x(t)|$$

first check that it is sufficient to prove that for any $\rho<1$, and $a>\sqrt{2}$

$$\sum_n P\big[\Delta_{\rho^n}(\omega)\ge a\sqrt{n\rho^n\log\frac{1}{\rho}}\big]<\infty \tag{2.5}$$

To estimate $\Delta_{\rho^n}(\omega)$ it is sufficient to estimate $\sup_{t\in I_j}|x(t)-x(t_j)|$ for $k_\epsilon\rho^{-n}$ overlapping intervals $\{I_j\}$ of the form $[t_j, t_j+(1+\epsilon)\rho^n\,]$ with length $(1+\epsilon)\rho^n$. For each $\epsilon>0$, $k_\epsilon=\epsilon^{-1}$ is a constant such that any interval $[s\,,t]$ of length no larger than ρ^n is completely contained in some I_j with $t_j\le s\le t_j+\epsilon\rho^n$. Then

$$\Delta_{\rho^n}(\omega)\le\sup_j\big[\sup_{t\in I_j}|x(t)-x(t_j)|+\sup_{t_j\le s\le t_j+\epsilon\rho^n}|x(s)-x(t_j)|\big]$$

Therefore, for any $a=a_1+a_2$,

$$\begin{aligned}
P&\Big[\Delta_{\rho^n}(\omega)\ge a\sqrt{n\rho^n\log\frac{1}{\rho}}\Big]\\
&\le P\Big[\sup_j\sup_{t\in I_j}|x(t)-x(t_j)|\ge a_1\sqrt{n\rho^n\log\frac{1}{\rho}}\Big]\\
&\quad + P\Big[\sup_j\sup_{t_j\le s\le t_j+\epsilon\rho^n}|x(s)-x(t_j)|\ge a_2\sqrt{n\rho^n\log\frac{1}{\rho}}\Big]\\
&\le 2k_\epsilon\rho^{-n}\Big[\exp[-\frac{a_1^2\,n\rho^n\log\frac{1}{\rho}}{2(1+\epsilon)\rho^n}]+\exp[-\frac{a_2^2\,n\rho^n\log\frac{1}{\rho}}{2\epsilon\rho^n}]\Big]
\end{aligned}$$

Since $a>\sqrt{2}$, we can pick $a_1>\sqrt{2}$ and $a_2>0$. For $\epsilon>0$ sufficiently small (2.5) is easily verified.