Chapter 11

Uniqueness: 1d

We will consider a one dimensional diffusion with $b(t,x) = 0$ and $0 < c \leq a(t,x) \leq C < \infty$. We want to prove that for any (s,x) there exists a unique process $P_{s,x}$ such that $P_{s,x} \in \mathcal{I}(a,0)$. It would then follow from Girsanov's theorem that the same is true for $[a,b]$ as well provided $b(s,x)$ is bounded. Since $a(s,x)$ need not be continuous we have to show existence as well. The proof depends on an estimate. Let us assume that $a(s,x)$ is Lipschitz in x. Then $\sigma(s,x)$ is Lipschitz as well and we do have a unique family $\{P_{s,x}\}$. We will approximate $a(s,x)$ by $a_n(s,x)$ and assume that $0 < c \leq a_n(s,x) \leq C$ and $a_n(s,x) \to a_(s,x)$ for almost all (s,x) w.r.t. Lebesgue measure on R^2. We verify the following:

1. The family $P^n_{s,x}$ that corresponds to $a_n(s,x)$ is uniformly tight. This comes easily from Kolmogorov's theorem because

$$E^{P^N_{s,x}}[|x(t) - x(s)|^4] \leq C|t-s|^2$$

with a constant C independent of n.

Fix s_0, x_0. Let a subsequence of $P^n_{s_0,x_0}$ converge weakly to P. Then it easy to verify that $P[x(s_0) = x_0] = 1$. But it is not obvious why

$$f(x(t)) - f(x_0) - \frac{1}{2}\int_{s_0}^{t} a(s,x(s))f''(x(s))ds$$

is a martingale. We need to show that if $g_n$is uniformly bounded and $g_n(s,x) \to g(s,x)$ a.e. Lebesgue on R^2, then

$$\lim_{n\to\infty} E^{P^n_{s_0,x_0}}[\int_{s_0}^{t} H(x(\cdot))g_n(s,x(s))ds] = E^P[\int_{s_0}^{t} H(x(\cdot))g(s,x(s))ds]$$

Where H is a bounded continuous function on $C[s_0,T]$. In such a case we can take the limits on both sides of

$$E^{P^n_{s_0,x_0}}\big[H(\omega)[f(x(t_2)) - f(x_0) - \frac{1}{2}\int_{s_0}^{t_2} a_n(s,x(s))f''(x(s))ds]\big]$$
$$= E^{P^n_{s_0,x_0}}\big[H(\omega)[f(x(t_2)) - f(x_0) - \frac{1}{2}\int_{s_0}^{t_1} a_n(s,x(s))f''(x(s))ds]\big]$$

to get

$$E^{P}\big[H(\omega)[f(x(t_2)) - f(x_0) - \frac{1}{2}\int_{s_0}^{t_2} a(s,x(s))f''(x(s))ds]\big]$$
$$= E^{P}\big[H(\omega)[f(x(t_2)) - f(x_0) - \frac{1}{2}\int_{s_0}^{t_1} a(s,x(s))f''(x(s))ds]\big]$$

for every bounded continuous $\mathcal{F}_{t_1}$ measurable function. If we define the linear functional

$$\Lambda_n(g) = E^{P^n_{s_0,x_0}}[H(\omega)\int_{s_0}^{T} g(s,x(s))ds]$$

and establish a uniform bound of the form

$$|\Lambda_n(g)| \le C(T)\|g\|_2 \tag{11.1}$$

that will be sufficient. If we represent

$$\Lambda_n(g) = \int_{[s_0,T]\times R} g(t,x)\lambda_n(dt,dx)$$

from the weak convergence of the processes $P^n_{s_0,x_0}$ we can conclude that the contribution to $\Lambda_n(g)$ comes mainly from a compact set $[s_0,T]\times[-A,A]$. From the estimate (9.1) it follows that $\lambda_n(dt,dx) = \lambda_n(t,x)dtdx$ and there is a uniform bound $\int|\lambda_n(t,x)|^2 dtdx \le C$. In particular we can assume that $\lambda_n(t,x)$ converges weakly in $L_2[[s_0,T]\times R$ to a limit $\lambda(t,x)$ and of course from the weak convergence of $P^n_{s_0,x_0}$ to P, assuming $H(\omega)$ to be bounded and continuous, it follows that

$$\int_{[s_0,T]\times R} g(t,x)\lambda(t,x)dtdx = E^P[H(\omega)\int_{s_0}^{T} g(s,x(s))ds]$$

If g_n are uniformly bounded and converge to g for almost all (t,x), then $g_n \to g$ in $L_2[[s_0,T]\times[-A,A]]$ and with the weak convergence of $\lambda_n \to \lambda$ it follows that

$$\lim_{n\to\infty}\int_{[s_0,T]\times[-A,A]} g_n(t,x)\lambda_n(t,x)dtdx = \int_{[s_0,T]\times[-A,A]} g(t,x)\lambda(t,x)dtdx$$

and the contribution to the integral from $[s_0,T]\times[-A,A]^c$ is uniformly small. We need the following result from PDE.

Theorem 11.1. *Consider* $[s_0, T] \times R$ *and functions* $u(t,x)$ *that are* C^∞ *and have compact support in* $[s_0, T) \times R$. *Let* $\mathcal{A}$ *be the operator*

$$(\mathcal{A}u)(t,x) = u_t + \frac{1}{2}a(t,x)u_{xx}(t,x)$$

with $0 < c \le a(t,x) \le C < \infty$. *Then the range of* $\mathcal{A}$ *is dense in* $L_2[[s_0,T] \times R]$ *and* $\mathcal{A}$ *is invertible with a bounded inverse* $u = Gf$, *that is bounded and continuous on* $[s_0, T] \times R$, $u(T,x) \equiv 0$ *and satisfies*

$$\sup_{\substack{t \in [s_0, T] \\ x \in R}} |u(t,x)| \le C_1[\|u_t\|_2 + \|u_{xx}\|_2] \le C_2\|f\|_2$$

with C_1, C_2 *depending only on* c *and* C.

Proof. We consider the operator

$$u(s,x) = (G_0 f)(s,x) = \int_s^T \int_R \frac{1}{\sqrt{2\pi(t-s)C}} e^{-\frac{(y-x)^2}{2C(t-s)}} f(y,t)dtdy$$

It is easy to check that u satisfies $u(T,x) \equiv 0$ and

$$u_s + \frac{C}{2}u_{xx} = -f$$

Moreover

$$|u(s,x)| \le K\|f\|_2$$

where

$$K^2 = \int_s^T \frac{1}{2\pi C(t-s)} e^{-\frac{(y-x)^2}{C(t-s)}} dydt = k\sqrt{C(T-s)}$$

We can solve explicitly by using Fourier transforms

$$\hat{u}(\tau,\xi) = \frac{\hat{f}}{\frac{1}{2}C\xi^2 - i\tau}$$

and estimate

$$\|\widehat{u_{xx}}\|_2 = \sup_{\tau,\xi} \Big| \frac{\xi^2}{\frac{1}{2}C\xi^2 - i\tau} \Big| \|\hat{f}\|_2 \le \frac{2}{C}\|\hat{f}\|_2$$

If we treat the operator

$$\mathcal{A} = u_s + \frac{1}{2}a(s,x)u_{xx}$$

as a perturbation

$$\mathcal{A}u = \mathcal{A}_0 u + Eu = u_s + \frac{C}{2}u_{xx} + Eu$$

then

$$\|Eu\|_2 \le \frac{C-c}{2}\|u_{xx}\|_2 \le \frac{C-c}{C}\|\mathcal{A}_0 u\|_2 = \rho\|\mathcal{A}_0 u\|_2$$

where $\rho < 1$. The operator $\mathcal{A}$ can be inverted as

$$(\mathcal{A}_0 + E)^{-1} = \mathcal{A}_0^{-1}(I + E\mathcal{A}_0^{-1})^{-1} = \mathcal{A}_0^{-1}B$$

Since $\|E\mathcal{A}_0^{-1}\|_{L_2 \to L_2} \le \rho < 1$, B is a bounded operator from $L_2 \to L_2$. ∎

The next theorem guarantees that any solution P corresponding to any a with $0 < c \leq a(t,x) \leq C < \infty$ satisfies the bound

$$|E^P[\int_s^T f(s,x(s))ds]| \leq C_{T-s}\|f\|_2$$

with a constant C_{T-s} depending only on c, C and $T-s$.

Theorem 11.2. *Let $x(t) = x + \int_0^t \sigma(s,\omega)d\beta(s)$ be a stochastic integral with $0 < c \leq \sigma^2(s,\omega) \leq C < \infty$. Then*

$$|E[\int_0^T f(s,x(s))ds]| \leq C_T\|f\|_2$$

Proof. If we prove it for simple σ then since the estimate is uniform, we can approximate the given σ by simple σ_n. Since simple ones are essentially piecewise constant and the estimate is true for Brownian motion, it is easy to verify that with

$$x_n(t) = x + \int_0^t \sigma_n(s,\omega)d\beta(s)$$

we do have

$$|E^P[\int_0^T f(s,x_n(s))ds]| \leq C_n\|f\|_2$$

for some finite C_n. It remains to bound C_n independent of n. □

If we take as before $u = G_0 f$ and apply Itô's formula

$$\begin{aligned} u(s,x) &= E^P\Big[\int_s^T [u_s(s,x_n(s)) + \frac{1}{2}a_n(s,\omega)u_{xx}(s,x_n(s))]ds\Big] \\ &= E^P\big[\int_s^T f(s,x_n(s)) + \frac{1}{2}[a_n(s,\omega) - C]u_{xx}(s,x_n(s))ds\big] \end{aligned}$$

We can conclude that

$$|E^P\big[\int_s^T f(s,x_n(s))ds\big]| \leq |u(s,x)| + \frac{C-c}{2}E^P\big[\int_s^T |u_{xx}(s,x_n(s))|ds\big]$$

If we use the bound $\|u_{xx}\|_2 \leq \frac{2}{C}\|f\|_2$, denoting by C_n the supremum

$$C_n = \sup_{\|f\|\leq 1} E^P\big[\int_s^T f(s,x_n(s))ds\big]|$$

we obtain

$$C_n \leq C_T + \frac{C-c}{2}\frac{2}{C}C_n = C_T + \rho C_n$$

Since $C_n < \infty$ we have $C_n \leq \frac{C_T}{1-\rho}$.

Remark 11.1. If $d = 1$ for any $[a,b]$ with $0 < c \leq a(s,x) \leq C < \infty$ and $|b| \leq C$ there is a unique $P = P_{s,x} \in \mathcal{I}(s,x,a,b)$. It is then a strong Markov process.