Chapter 10

Uniqueness

It is clear from previous discussions that it is important to establish general conditions under which uniqueness holds. Uniqueness implies Markov and strong Markov properties of the family $\{P_{s,x}\}$. It will also imply that if the coefficients are independent of t, i.e. depend only on x, then the transition probabilities $p(s,x,t,A)$ are stationary and depend only on $(t-s)$. We have the following cases where we have established uniqueness.

1. If $a = \sigma\sigma^*$ and σ, b are uniformly bounded and Lipschitz in x, i.e.

$$|\sigma(t,x) - \sigma(t,y)| + |b(t,x) - b(t,y)| \leq C|x-y|$$

and

$$|\sigma(t,x)| + |b(t,x)| \leq C$$

implies path wise uniqueness and therefore distribution uniqueness.

2. If we have uniqueness when $b = 0$ and a is uniformly elliptic i.e.

$$c\sum_i \xi_i^2 \leq \sum_{i,j} a_{i,j}(t,x)\xi_i\xi_j \leq C\sum_i \xi_i^2$$

for some $0 < c \leq C < \infty$, and $|b(t,x)| \leq C$, then by Girsanov's theorem, uniqueness holds for $[a,b]$.

3. If the PDE

$$u_s + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)(D_{x_i}D_{x_j}u)(s,x) + \sum b_j(s,x)(D_{x_j}u)(s,x) = 0; s < t$$

with $u(s,x) \to f(x)$as $s \uparrow t$ has "smooth" solutions for sufficiently many f, then uniqueness holds for $[a,b]$. The proof uses Itô's formula and requires u to be bounded and have one continuous t derivative and two continuous x derivatives. We can then conclude that if $P \in \mathcal{I}(s_0, x_0, a, b)$ is any solution, then $u(s, x(s))$ is a martingale and equating expectations at time s_0 and t we get

$$E^P[f(x(t)] = u(s_0, x_0)$$

If this is valid for any (s_0, x_0) sufficiently many f, then $\mu_{s,x,t}(A)$ defined is defined uniquely by

$$u(s,x) = \int f(y)\mu_{s,x,t}(dy)$$

From the fact that the conditional probabilities $P|\mathcal{F}_t$ are again solutions it follows that for $s < t_1 < t_2$

$$P_{s,x}[X(t_2) \in A|\mathcal{F}_{t_1}] = P_{t_1,x(t_1)}[x(t_2) \in A] = p(t_1, x(t_1), t_2, A)$$

proving the Markov property as well as determining the transition probability.

4. In the time homogeneous case one can use the resolvent equation,

$$\lambda u - \frac{1}{2}\sum_{i,j} a_{i,j}(x)(D_{x_i}D_{x_j}u)(x) + \sum b_j(x)(D_{x_j}u)(x) = f(x)$$

If $u(x)$ is a bounded twice continuously differentiable solution of the above equation then by Itô's formula

$$Z(t) = e^{-\lambda t}u(x(t)) + \int_0^t f(x(s))ds$$

will be a martingale for any solution P_x starting from $(0, x)$. Equating expectations at $t = 0$ and $t = +\infty$, we get

$$u(x) = \int_0^\infty e^{-\lambda s}E[f(x(s))]ds$$

This will determine uniquely the stationary transition probabilities $p(t, x, A)$ as $P_x[x(t) \in A]$ and therefore the entire process.

5. We can try and solve the parabolic equation

$$u_s + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)(D_{x_i}D_{x_j}u)(s,x) + \sum b_j(s,x)(D_{x_j}u)(s,x) + f(s,x); s < t$$

and $u(t, x) = 0$. If we have a classical solution then

$$u(s, x(s)) + \int_{s_0}^s f(v, x(v))dv$$

is a martingale and we get

$$u(s_0, x_0) = E[\int_{s_0}^t f(s, x(s))ds]$$

If we can claim this for sufficiently many functions that would prove uniqueness as well.

6. Finally if we can show that these solutions exist in some generalized sense, then we can still succeed in proving uniqueness provided we can show, by apriori estimates, that objects like

$$\Lambda(f) = E[\int_{s_0}^{t} f(s, x(s))ds]$$

are in the appropriate dual space.