Chapter 3

Riesz transforms on $R^d$

3.1 Fourier Integrals.

We now look at Fourier Transforms on $R^d$. If $f(x)$ is a function in $L_1(R^d)$ its Fourier transform $\hat{f}(y)$ is defined by

$$\hat{f}(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i<x,y>} f(x)dx$$

(3.1)

We denote by $S$ the class of all functions $f$ on $R^d$ that are infinitely differentiable such that the function and its derivatives of all orders decay faster than any power, i.e. for every $n_1, n_2, \ldots, n_d \geq 0$ and $k \geq 0$ there are constants $C_{n_1, n_2, \ldots, n_d, k}$ such that

$$||[(\frac{d}{dx_1})^{n_1}(\frac{d}{dx_1})^{n_2}\ldots(\frac{d}{dx_d})^{n_d}f](x)|| \leq C_{n_1, n_2, \ldots, n_d, k}(1 + ||x||)^{-k}$$

It is easy to show (left as an exercise) by repeated integration by parts that if $f \in S$ so does $\hat{f}$.

Theorem 3.1. The Fourier transform has the inverse

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \hat{f}(y)dy$$

(3.2)

proving that the Fourier transform is a one to one mapping of $S$ onto itself.

In addition the Fourier transform extends as a unitary map from $L_2(R^d)$ onto $L_2(R^d)$. 
Proof. Clearly
\[ g(x) = \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} \hat{f}(y) dy \]
is well defined as a function in $S$. We only have to identify it. We compute $g$ as

\[
g(x) = \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} \hat{f}(y) dy \\
= \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} \hat{f}(y) e^{-\epsilon \|y\|^2 / 2} dy \\
= \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} e^{i\langle z, y \rangle} f(z) dz \right] e^{-i\langle x, y \rangle} e^{-\epsilon \|y\|^2 / 2} dy \\
= \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle z-x, y \rangle} f(z) e^{-\epsilon \|y\|^2 / 2} dy dz \\
= \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} f(z) \left[ \int_{\mathbb{R}^d} e^{i\langle z-x, y \rangle} e^{-\epsilon \|y\|^2 / 2} dy \right] dz \\
= \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi\epsilon}} \right)^d \int_{\mathbb{R}^d} f(z) e^{-\frac{\|z-x\|^2}{2\epsilon}} dz \\
= f(x)
\]

Here we have used the identity
\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i xy} e^{-x^2 / 2} dx = e^{-y^2 / 2}
\]

We now turn to the computation of $L_2$ norm of $\hat{f}$. We calculate it as
\[ \| \hat{f} \|_2^2 = \lim_{\epsilon \to 0} \int_{R^d} |\hat{f}(y)|^2 e^{-\frac{\|y\|^2}{2}} dy \]
\[ = \lim_{\epsilon \to 0} \int_{R^d} \int_{R^d} f(x) \bar{f}(z) e^{i <x-z,y>} e^{-\frac{\|y\|^2}{2}} dy dx dz \]
\[ = \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{2\pi \epsilon}} \right)^d \int_{R^d} \int_{R^d} f(x) \bar{f}(z) e^{-\frac{\|x-z\|^2}{2\epsilon}} dx dz \]
\[ = \lim_{\epsilon \to 0} \int_{R^d} f(x) \left[ K_{\epsilon} \hat{f} \right](x) dx \]
\[ = \int_{R^d} |f(x)|^2 dx \]

Since the \( f \to \hat{f} \) preserves the \( L_2 \) norm and is onto from \( S \to S \), it extends to the completion \( L_2 \) as a unitary map. \( \square \)

We see that the Fourier transform is a bounded linear map from \( L_1 \) to \( L_\infty \) as well as \( L_2 \) to \( L_2 \) with corresponding bounds \( C = (\frac{1}{\sqrt{2\pi \epsilon}})^d \) and 1. By the Riesz-Thorin interpolation theorem (see the exercise in Chapter 2) the Fourier transform is bounded from \( L_p \) into \( L_{\frac{p}{p-1}} \) for \( 1 \leq p \leq 2 \). If \( \frac{1}{p} = 1.1 + \frac{1}{2}(1-t) \) then \( \frac{1}{p} = 1 - \frac{1}{p} = \frac{p-1}{p} \). See exercise to show that, for \( f \in L_p \) with \( p > 2 \), the Fourier Transform need not exist.

For convolution operators of the form
\[ (Tf)(x) = (k \ast f)(x) = \int_{R^d} k(x-y) f(y) dy \] (3.3)
we want to estimate \( \|T\|_p \), the operator norm from \( L_p \) to \( L_p \) for \( 1 \leq p \leq \infty \). As before for \( p = 1, \infty \),
\[ \|T\|_p = \int_{R^d} |k(y)| dy. \]
Let us suppose that for some constant \( C \),

1. The Fourier transform \( \hat{k}(y) \) of \( k(\cdot) \) satisfies
\[ \sup_{y \in R^d} |\hat{k}(y)| \leq C < \infty \] (3.4)
2. In addition,
\[
\sup_{x \in \mathbb{R}^d} \int_{\{y : \|x-y\| \geq C\|x\|\}} |k(y - x) - k(y)| dy \leq C < \infty \quad (3.5)
\]

We will estimate \(\|T\|_p\) in terms of \(C\). The main step is to establish a weak type \((1,1)\) inequality. Then we will use the interpolation theorems to get boundedness in the range \(1 < p \leq 2\) and duality to reach the interval \(2 \leq p < \infty\).

**Theorem 3.2.** The function \(g(x) = (Tf)(x) = (k * f)(x)\) satisfies a weak type \((1,1)\) inequality
\[
\mu\{x : |g(x)| \geq \ell\} \leq C_0 \frac{\|f\|_1}{\ell} \quad (3.6)
\]
with a constant \(C_0\) that depends only on \(C\).

We first prove a decomposition lemma that we will need for the proof of the theorem.

**Lemma 3.3.** Given any open set \(G \in \mathbb{R}^d\) of finite Lebesgue measure we can find a countable set of balls \(\{S(x_j, r_j)\}\) with the following properties. The balls are all disjoint. \(G = \bigcup_j S(x_j, 3r_j)\) is the countable union of balls with the same centers but three times the radius. Moreover there is a number \(k_1(d)\) that depends only on the dimension such that each point of \(G\) is covered at most \((96)^d\) times by the covering \(G = \bigcup_j S(x_j, 3r_j)\). Finally each of the balls \(S(x_j, 5r_j)\) has a nonempty intersection with \(G^c\).

Basically, the lemma says that it is possible to write \(G\) as a nearly disjoint countable union of balls each having a radius that is comparable to the distance of its center to the boundary.

**Proof.** Suppose \(G\) is an open set in the plane of finite volume.

Let \(d(x) = d(x, G^c)\) be the distance from \(x\) to \(G^c\) or the boundary of \(G\). Let \(d_0 = \sup_{x \in G} d(x)\). Since the volume of \(G\) is finite, \(G\) cannot contain any large balls and consequently \(d_0\) cannot be infinite.

We consider balls \(S(x, r(x))\) around \(x\) of radius \(r(x) = \frac{d(x)}{4}\). They are contained in \(G\) and provide a covering of \(G\) as \(x\) varies over \(G\). All these balls have the property that \(S(x, 3r(x)) \subset G\) and \(S(x, 5r(x))\) intersects \(G^c\).
3.1. FOURIER INTEGRALS.

We proceed to select a countable collection \( \{S(x_i, r(x_i))\} \) from \( \{S(x, r(x))\} \) that are disjoint while \( \cup_i S(x_i, 3r(x_i)) = G \).

We choose \( x_1 \) such that \( d(x_1) > \frac{d_k}{2} \). Having chosen \( x_1, \ldots, x_k \) the choice of \( x_{k+1} \) is made as follows. We consider the balls \( S(x_i, r(x_i)) \) for \( i = 1, 2, \ldots, k \). Look at the set \( G_k = \{ x : S(x, r(x)) \cap S(x_i, r(x_i)) = \emptyset \text{ for } 1 \leq i \leq k \} \) and define \( d_k = \sup_{x \in G_k} d(x) \). We pick \( x_{k+1} \in G_k \) such that \( d(x_{k+1}) > \frac{d_k}{2} \). We proceed in this fashion to get a countable collection of balls \( \{S(x_i, r(x_i))\} \).

By construction, they are disjoint balls contained in the set \( G \) of finite volume and therefore \( r(x_i) \to 0 \) as \( i \to \infty \). Since, \( d_i \leq 2d(x_{i+1}) \leq 8r(x_{i+1}) \) it follows that \( d_i \to 0 \) as \( i \to \infty \). Every \( S(x_i, 5r(x_i)) \) intersects \( G^c \).

We now examine how much of \( G \) the balls \( \{B(x_i, r(x_i))\} \) cover. First we note that

\[
G_0 \supset G_1 \supset \cdots \supset G_k \supset G_{k+1} \supset \cdots
\]

We claim that \( \cap_k G_k = \emptyset \). If not, let \( x \in G_k \) for every \( k \). Then \( d_k = \sup_{y \in G_k} d(y) \geq d(x) > 0 \) for every \( k \). This contradicts the convergence of \( d_k \) to 0.

Since \( x \in G_0 = G \), we can find \( k \geq 1 \) such that \( x \in G_{k-1} \) but \( x \notin G_k \). Then \( S(x, r(x)) \) must intersect \( S(x_k, r(x_k)) \) giving us the inequality

\[
|x-x_k| < r(x)+r(x_k) \leq \frac{d(x)}{4}+r(x_k) \leq \frac{d_{k-1}}{4}+r(x_k) \leq \frac{d(x_k)}{2}+r(x_k) = 3r(x_k)
\]

Clearly \( S(x_k, 3r(x_k)) \) will contain \( x \). Since \( 3r(x) < d(x) \) the enlarged ball is still within \( G \). This means \( G = \cup_k S(x_k, 3r(x_k)) \).

Now we will bound the number of times a point \( x \) can be covered by \( \{S(x_k, 3r(x_k))\} \). Let for some \( k \), \( |x-x_k| < 3r(x_k) \). Then by the triangle inequality

\[
|d(x) - d(x_k)| \leq 3r(x_k)
\]

or equivalently (recall \( r(x) = \frac{d(x)}{4} \))

\[
|r(x) - r(x_k)| \leq \frac{3}{4} r(x_k)
\]

This implies that for the ratio \( |\frac{r(x)}{r(x_k)} - 1| \leq \frac{3}{4} \) we have \( \frac{1}{4} \leq \frac{r(x)}{r(x_k)} \leq \frac{7}{4} \). In particular any ball \( S(x_k, 3r(x_k)) \) that covers \( x \), must have its center with in a distance of \( 3r(x_k) \leq 12r(x) \) of \( x \) and the corresponding \( r(x_k) \) must be in
the range $\frac{4}{7}r(x) \leq r(x_k) \leq 4r(x)$. The balls $S(x_k, r(x_k))$ are then contained in $S(x, 24r(x))$ are disjoint and have a radius of at least $\frac{4}{7}r(x)$. There can be at most $k_1(d) = (42)^d$ of them by considering the total volume. We can choose our norm in $\mathbb{R}^d$ to be $\max_i |x_i|$ and force the spheres to be cubes.

$\square$

**Proof of theorem.** The proof is similar to the one-dimensional case with some modifications.

1. We let $G_\ell$ be the open set where the maximal function $Mf(x)$ satisfies $|Mf(x)| > \ell$. From the maximal inequality

$$\mu[G_\ell] \leq C(d) \frac{\|f\|_1}{\ell} \quad (3.7)$$

2. We write $G_\ell = \bigcup_i B_i = \bigcup_i S(x_i, 3r(x_i))$, a countable union of spheres according to the lemma.

3. If we let

$$\phi(x) = \sum_i 1_{B_i}(x)$$

then $1 \leq \phi(x) \leq k_1(d)$ on $G_\ell$.

4. Let us define a weighted average $m_i$ of $f(y)$ on $B_i$ by

$$\int_{B_i} [f(y) - m_i] \frac{dy}{\phi(y)} = 0 \quad (3.8)$$

and write

$$f(x) = f(x) 1_{G_\ell}(x) + \frac{1}{\phi(x)} \sum_i f(x) 1_{B_i}(x)$$

$$= \left[ f(x) 1_{G_\ell}(x) + \frac{1}{\phi(x)} \sum_i m_i 1_{B_i}(x) \right]$$

$$+ \frac{1}{\phi(x)} \sum_i [f(x) - m_i] 1_{B_i}(x)$$

$$= h_0(x) + \sum_i h_i(x) \quad (3.10)$$
3.1. FOURIER INTEGRALS.

5. For any sphere $B_i$ with center $x_i$ and radius $3r(x_i)$ there is a sphere with radius $5r(x_i)$ with the same center that contains a point $x'_i \in G_\ell^c$ with $|M_f(x'_i)| \leq \ell$. The sphere $\hat{B}_i = S(x'_i, 8r(x_i))$ contains $B_i$. Since $1 \leq \phi(y) \leq k_1(d)$ on $G_\ell$

$$|m_i| \leq \left[ \int_{B_i} \frac{|f(y)|}{\phi(y)} dy \right] \left[ \int_{B_i} \frac{1}{\phi(y)} dy \right]^{-1}$$

$$\leq k_1(d) \frac{1}{\mu(B_i)} \int_{B_i} |f(y)| dy$$

$$= k_1(d) \left( \frac{8}{3} \right)^d \frac{1}{\mu(\hat{B}_i)} \int_{\hat{B}_i} |f(y)| dy \leq k_2(d)$$

$$\leq k_2(d) \frac{1}{\mu(\hat{B}_i)} \int_{\hat{B}_i} |f(y)| dy$$

$$\leq k_2(d) M_f(x'_i)$$

$$\leq k_2(d) \ell$$

It follows that on $G_\ell$

$$\frac{1}{\phi(x)} \sum_i m_i 1_{B_i}(x) \leq k_2(d) \ell$$

Moreover on $G_\ell^c$, $|f(x)| \leq M_f(x) \leq \ell$. Since $k_2(d) \geq 1$

$$\|h_0\|_{\infty} \leq \max\{1, k_2(d)\} \ell = k_2(d) \ell$$

(3.11)

On the other hand $\phi(x) \geq 1$ on $G_\ell$ and

$$\|h_0\|_1 \leq \|f\|_1 + k_2(d) \ell \sum_i \mu[B_i]$$

$$\leq \|f\|_1 + k_2(d) \ell \mu[G_\ell]$$

$$\leq (1 + k_2(d)) \|f\|_1$$

(3.12)

and therefore

$$\|h_0\|_2^2 \leq \|h_0\|_1 \|h_0\|_{\infty} \leq k_3(d) \ell \|f\|_1$$

(3.13)

From the boundedness of $T$ from $L_2$ to $L_2$ this gives

$$\mu\{x : |(Th_0)(x)| \geq \ell\} \leq \frac{\|Th_0\|_2^2}{\ell^2} \leq C^2 k_3(d) \|f\|_1$$

(3.14)

where $C$ is the bound on $|\hat{k}|$ from (3.4)
6. We now turn our attention to the functions \( \{h_j\} \)

\[
w(x) = \sum_i (Th_i)(x) = \sum_i \int_{B_i} [f(y) - m_j]k(x - y) \frac{dy}{\phi(y)}
= \sum_i \int_{B_i} [f(y) - m_i]k(x - y) - k(x - x_i) \frac{dy}{\phi(y)}
\]

\[
|w(x)| \leq \sum_i \int_{B_i} |f(y) - m_i||k(x - y) - k(x - x_i)|dy
\tag{3.15}
\]

We estimate \( |w(x)| \) for \( x \notin \bigcup_i U_i \) where \( U_i \) is the sphere with the same center \( x_i \) as \( B_i \) but enlarged by a factor \( C + 1 \). In particular if \( y \in B_i \) and \( x \in U_i \), then \( |y - x| \geq |x - x_i| - |y - x_i| \geq C|y - x_i| \).

\[
\int_{\cap_i U_i} |w(x)|dx \leq \sum_i \int_{\cap_i U_i} \left[ \int_{B_i} |f(y) - m_i||k(x - y) - k(x - x_i)|dy\right]dx
\leq \sum_i \int_{B_i} |f(y) - m_i|\left[ \int_{E_i} |k(x - y) - k(x - x_i)|dx\right]dy
\tag{3.16}
\]

where \( E_i \subset \{x : |x - y| \geq C|y - x_i|\} \). Therefore,

\[
\int_{E_i} |k(x - y) - k(x - x_i)|dx
\leq \sup_{y,i} \int_{\{x : |x - y| \geq C|y - x_i|\}} |k(x - y) - k(x - x_i)|dx
\leq \sup_{y} \int_{\{x : |x - y| \geq C|y|\}} |k(x - y) - k(x)|dx
\leq C
\tag{3.17}
\]

giving us the estimate
3.1. FOURIER INTEGRALS.

\[
\int_{\cap_i U_i} |w(x)| dx \leq C \sum_i \int_{B_i} |f(y) - m_i| dy \\
\leq C(\|f\|_1 + \sup_m \sum_i \mu[B_i]) \\
\leq C[\|f\|_1 + k_2(d)\ell\mu(G_\ell)] \tag{3.18} \\
\leq k_3(d)\|f\|_1 \tag{3.19}
\]

7. We can estimate \(\mu(\cup_i U_i) \leq \sum_i \mu(U_i)\) by

\[
\sum_i \mu(U_i) \leq k_4(d) \sum_i \mu(B_i) \leq k_5(d)\mu(G_\ell) \leq k_6(d)\frac{\|f\|_1}{\ell}
\]

8. We put the pieces together and we are done.

\[\square\]