One way to construct a diffusion process corresponding to the operator
\[(Lf)(x) = \frac{1}{2} \sum_{i,j} \partial^2 u \left( \frac{\partial u}{\partial x_i \partial x_j} \right) + \sum_j b_j(x) \frac{\partial u}{\partial x_j}(x)\]
is to find a process with the property
\[x(t + h) - x(t) \simeq \sqrt{h}Z + hb(x(t))\]
where \(Z\) is a Gaussian with dispersion \(a_{i,j}(x(t))\). If \(\sigma(x)\sigma^*(x) = a(x)\), \(\sigma(x)\) maps \(R^k \rightarrow R^d\) then \(Z = \sigma(x)[\beta(t + h) - \beta(t)]\) should work. This leads to
\[dx(t) = \sigma(x(t)) \cdot d\beta(t) + b(x(t))dt\]

There are three possible equivalent formulations of what a solution to the equation means.

For any \(x \in R^d\), there is a measure \(P\) on \(\Omega = C[[0, T]; R^d]\) such that \(P[x(0) = 0] = 1\), and for any smooth \(f\) with compact support on \(R^d\)
\[f(x(t)) - f(x) - \int_0^t (Lf)(x(s))ds\]
is a martingale with respect to \((\Omega, \mathcal{F}_t, P)\).

Or equivalently there is a measure space and a filtration \((\Omega, \mathcal{F}_t, P)\) and two progressively measurable almost surely continuous processes \(x(t, \omega), \beta(t, \omega)\) with values in \(R^d\) and \(R^k\) respectively, where \(\beta(t, \omega)\) is a \(k\)-dimensional Brownian motion adapted to \(\mathcal{F}_t\), i.e. for any \(t > s\), \(\beta(t) - \beta(s)\) is independent of \(\mathcal{F}_s\). They satisfy
\[x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds\]

This can be rephrased as finding a measure \(Q\) on \((C[[0, T], R^d \times R^k), \mathcal{F}_t, Q)\) such that for smooth \(f\) with compact support on \(R^d \times R^k\)
\[f(x(t), y(t)) - f(x(0), 0) - \int_0^t (Lf)(x(s), y(s))ds\]
is a martingale, where
\[(Lf)(x, y) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x, y) + \sum_{j=1}^d b_j(x) \frac{\partial f}{\partial x_j}(x, y)\]
\[+ \frac{1}{2} \sum_{i=1}^k \frac{\partial^2 f}{\partial y_i^2}(x, y) + \sum_{i=1}^d \sum_{j=1}^k \sigma_{i,j}(x, y) \frac{\partial f}{\partial x_i \partial y_j}(x, y)\]
Or one can ask if on the canonical Brownian motion space \((C[[0,T],R^k),\mathcal{F}_t,P)\) the equation

\[
x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\omega(s) + \int_0^t b(x(s))ds
\]

can be solved with a progressively measurable almost surely continuous solution \(x(t,\omega)\).

If \(\sigma\) satisfies the Lipschitz condition \(\|\sigma(x) - \sigma(y)\| \leq C|x-y|\), then for any initial random variable \(\xi\) measurable w.r.t. \(\mathcal{F}_0\) with \(\|\xi\|_2 < \infty\) the above equation has a unique solution.

**Existence.** Let us define recursively starting with \(x_0(t) \equiv \xi\)

\[
x_{n+1}(t) = \xi + \int_0^t \sigma(x_n(s)) \cdot d\beta(s) + \int_0^t b(x_n(s))ds
\]

Inductively \(\sigma_n\) is progressively measurable and bounded. Hence so is \(x_n(t)\). Taking the difference

\[
x_{n+1}(t) - x_n(t) = \int_0^t [\sigma(x_n(s)) - \sigma(x_{n-1}(s))] \cdot d\beta(s) + \int_0^t [b(x_n(s)) - b(x_{n-1}(s))]ds
\]

Let us denote by \(\Delta_n(t) = E[\sup_{0 \leq s \leq t} \|x_{n+1}(s) - x_n(s)\|^2]\). Then

\[
\Delta_n(t) \leq 2E[\sup_{0 \leq s \leq t} |\int_0^s [\sigma(x_n(\tau)) - \sigma(x_{n-1}(\tau))] \cdot d\beta(\tau)|^2 + |\int_0^t [b(x_n(s)) - b(x_{n-1}(s))]ds|^2]
\]

By Doob’s inequality the first term is dominated by \(8E[\int_0^t |\sigma(x_n(\tau)) - \sigma(x_{n-1}(\tau))|^2]\) and the second by \(2TE[|\int_0^t |b(x_n(s)) - b(x_{n-1}(s))|^2ds|^2]\). If we consider a finite interval \([0,T]\).

using the Lipschitz condition

\[
\Delta_{n+1}(t) \leq C(T) \int_0^t \Delta_n(s)ds
\]

with

\[
\Delta_0(t) = 8E[\|\sigma(\xi) \cdot (\beta(t) - \beta(0))\|^2 + 2T|b(\xi)|^2] \leq c(T)t
\]

By induction

\[
\Delta_n(t) \leq \frac{[C(T)]^{n+1}}{(n+1)!}
\]

Since \(\sum_n \sqrt{\Delta_n(t)} < \infty\), it follows that

\[
P\left[\sum_n \sup_{0 \leq t \leq T} \|x_{n+1}(t) - x_n(t)\| < \infty\right] = 1
\]
Therefore $\lim_{n \to \infty} x_n(t) = x(t)$ exists almost surely and passing to the limit
\[ x(t) = x(0) + \int_0^t \sigma(x(s)) \cdot d\omega(s) + \int_0^t b(x(s))ds \]

**Uniqueness.** For $i = 1, 2$
\[ x_i(t) = \xi + \int_0^t \sigma(x_i(s)) \cdot d\beta(s) + \int_0^t b(x_i(s))ds \]

Let $y(t) = x_1(t) - x_2(t)$ and $\delta(t) = E[\|y(t)\|^2]$.
\[ \delta(t) \leq C(T) \int_0^t \delta(s)ds \]

Implies $\delta(t) \equiv 0$.

**Markov and Strong Markov Property.**

If you start the solution from $x(0) = x$ and run it up to a stopping time $\tau$, then the solution starting from $x(\tau)$ is the same as the old one. But the Brownian increments after time $\tau$ are independent of $\mathcal{F}_\tau$. This is strong Markov property. The discrete analog is if $X_{n+1} = f(X_n, Y_{n+1})$ where $\{Y_n\}$ are mutually independent and independent of $X_0$, then $\{X_n\}$ is a Markov process.

If the SDE
\[ x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds \]
has a unique solution for some choice of $\sigma$ satisfying $\sigma(x)\sigma^*(x) = a(x)$ then the Markov family $\{P_x\}$ the distributions of $(\cdot)$ for the varying starting points $x \in R^d$, are solutions to the martingale problem for $L$. Does it imply that there are no other solutions to the Martingale Problem?

**Theorem.** If $P$ is any solution of the martingale problem and if $a(x) = \sigma(x)\sigma^*(x)$ for some choice of $\sigma$ for which the solution to the SDE is unique then the solution to the martingale problem is unique.

**Proof.** If $P_1$ and $P_2$ are two solutions to the martingale problem and $a(x) = \sigma(x)\sigma^*(x)$ then on the space $\Omega = C[[0, T]; R^d \times R^k]$ there are two probability measures $\hat{P}_1$ and $\hat{P}_2$. There projections on $C[[0, T]; R^d]$ are $P_1$ and $P_2$ while their projections on the second component $C[[0, T]; R^k]$ is the Brownian motion $\mu$. They are related by
\[ x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds \]

with $\omega \in \Omega$ being $\omega(t) = (x(t), \beta(t))$ If we can construct a measure $\hat{Q}$ on $\hat{\Omega} = C[[0, T]; R^d \times R^d \times R^k]$ with $\hat{\omega} = (x(t), y(t), \beta(t))$ such that $(x(\cdot), \beta(\cdot))$ and $(y(\cdot), \beta(\cdot))$ have distributions $\hat{P}_1$ and $\hat{P}_2$ then we would have
\[ x(t) = x + \int_0^t \sigma(x(s)) \cdot d\beta(s) + \int_0^t b(x(s))ds \]
as well as
\[ y(t) = x + \int_0^t \sigma(y(s)) \cdot d\beta(s) + \int_0^t b(y(s))ds \]
implying \( x(t) \equiv y(t) \) proving \( P_1 = P_2 \).

**Construction of \( \hat{Q} \).** Let us denote by \( q_1^\beta(d\omega_1) \) the regular conditional probability distribution of \( x(\cdot) \) given \( \beta(\cdot) \) and by \( q_2^\beta(d\omega_1) \) the regular conditional probability distribution of \( y(\cdot) \) given \( \beta(\cdot) \). We define \( \hat{Q} \) by
\[
Q(d\omega_1, d\omega_2, d\beta) = q_1^\beta(d\omega_1) \times q_2^\beta(d\omega_1) \times \mu(d\beta)
\]
We need to make sure that the Brownian increments after time \( t \) are independent of the \( \sigma \)-field \( \mathcal{F}_t \) generated by \( \{x(s), y(s), \beta(s)\} \) where \( 0 \leq s \leq t \). This is easily checked. Depends on the fact that the conditional distribution of \( x(\cdot), y(\cdot) \) on \([0, t]\) given \( \beta(\cdot) \) on \([0, T]\) depends only on \( \beta(\cdot) \) on \([0, t]\).

Given \( a(x) \) when can we find a \( \sigma(x) \) with \( \sigma(x)\sigma^*(x) = a(x) \)?