What do we mean by a stochastic process with continuous paths on \( R^d \) with characteristics \( \{a_{i,j}(t,\omega)\} \) and \( \{b_j(t,\omega)\} \) or solution to the martingale problem corresponding to \( \{a_{i,j}(t,\omega); \{b_j(t,\omega)\}\}? 

\( \Omega = C[[0,T];R^d] \) is the space of continuous \( R^d \) valued function on \([0,T]\). \( F_t \) is the \( \sigma \)-field generated by \( \{x(s)\}, 0 \leq s \leq t \). \( T \) can be finite or \( \infty \) in which case we have \([0,\infty)\) instead of \([0,T]\). \( \Omega = C[[0,T];R^d] \) is the space of continuous \( R^d \) valued function on \([0,T]\).

We say that \( P \) is a process with characteristics \( a, b \) with initial distribution \( \mu \) if \( P[x(0) \in A] = \mu(A) \) and any one of the following which is equivalent are true.

1. For any smooth function \( f \) with compact support on \( R^d \)
\[
f(t,x) - f(0,x) - \int_0^t (L_{s,\omega}f)(s,x)\,ds
\]
is a martingale with respect to \((\Omega,F_t,P)\) Here
\[
(L_{s,\omega}f)(s,x) = \frac{1}{2} \sum_{i,j} a_{i,j}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(s,x) + \sum_j b_j(s,\omega) \frac{\partial f}{\partial x_j}(s,x)
\]

2. For any function \( f(t,x) \) in \( C^{1,2}([0,T] \times R^d) \)
\[
f(t,x) - f(0,x) - \int_0^t [\frac{\partial f}{\partial s}(s,x) + (L_{s,\omega}f)(s,x)]\,ds
\]
is a martingale.

3. For any function \( f \) in \( C^{1,2}([0,T] \times R^d) \)
\[
\exp[f(t,x) - f(0,x) - \int_0^t [e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s,x)\,ds]
\]
is a martingale with respect to \((\Omega,F_t,P)\)

Remark: \( f \) and its derivatives can have growth \( o(|x|^2) \) at infinity. In particular
\[
P\left[ \sup_{0 \leq t \leq T} \|x(t,\omega)\| \geq \ell \right] \leq C(T) \exp[-c_0(T)\ell^2]
\]

4. For any \( \theta \in R^d \)
\[
\exp[<\theta,x(t,\omega) - x(0,\omega)> - \frac{1}{2} \int_0^t <a(s,\omega)\theta, \theta>\,ds - \int_0^t <b(s,\omega), \theta>\,ds]
\]
is a martingale.

**Proofs.** We can assume without loss of generality that \(f(t, x)\) is \(C^\infty([0, T] \times \mathbb{R}^d)\).

\[
E[f(t, x(t)) - f(s, x(s))|\mathcal{F}_s] = E[f(t, x(t)) - f(s, x(t)) + f(s, x(t)) - f(s, x(s))|\mathcal{F}_s]
\]

\[
= E[\int_s^t f_v(v, x(t))dv + \int_s^t (L_{v,\omega} f)(s, x(v))dv|\mathcal{F}_s]
\]

\[
= E[\int_s^t f_v(v, x(v))dv + \int_s^t dv \int_v^t L_{u,\omega} f_v(v, x(u))du
\]

\[
+ \int_s^t (L_{v,\omega} f)(v, x(v))dv - \int_s^v du \int_s^t (L_{v,\omega} f)(u, x(v))dv|\mathcal{F}_s]
\]

\[
= E[\int_s^t f_v(v, x(v))dv + \int_s^t (L_{v,\omega} f)(v, x(v))dv]
\]

**Lemma.** Let \(M(t)\) be a continuous martingale on \((\Omega, \mathcal{F}_t, P)\) and \(A(t)\) a progressively measurable continuous function of bounded variation with \(A(0) = 0\). Assume for any finite \(T\), \(M(T)\) is square integrable and the total variation \(|A|(T)\) of \(A(t)\) on \([0, T]\) is square integrable, then

\[
A(t)M(t) - \int_0^t M(s)dA(s)
\]

is a martingale with respect to \((\Omega, \mathcal{F}_t, P)\).

**Proof.**

\[
E[A(t)M(t) - A(s)M(s) - \int_s^t M(u)dA(u)|\mathcal{F}_s]
\]

\[
= \lim_{\pi \downarrow 0} \sum_j E[A(t_j)M(t_j) - A(t_{j-1})M(t_{j-1}) - \int_{t_{j-1}}^{t_j} M(u)dA(u)|\mathcal{F}_s]
\]

\[
= \lim_{\pi \downarrow 0} \sum_j E[A(t_j)M(t_j) - A(t_{j-1})M(t_j) - \int_{t_{j-1}}^{t_j} M(u)dA(u)|\mathcal{F}_s]
\]

\[
= 0
\]

To go from 2. to 3.

\[
M(t) = e^{f(t, x(t), \omega)} - \int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s, \omega))ds
\]

\[
A(t) = \exp[-f(0, x(0, \omega)) - \int_0^t [(e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f)(s, x(s, \omega))]ds]
\]
\[ M(t)A(t) - \int_0^t M(s)dA(s) \] simplifies to (3) because
\[ A(t) \int_0^t [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))ds + \int_0^t M(s)dA(s) = 0 \]

To verify this let us differentiate with respect to \( t \).
\[ A'(t) \int_0^t [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))ds + A(t)[\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega)) + A'(t)M(t) = 0 \]
\[ A'(t) = -A(t)[(e^{-f}(\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f)(s, x(s, \omega))] \]
\[ M(t) = e^{f(t,x(t,\omega))} - \int_0^t [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))ds \]

We see that after dividing by \( A(t) \)
\[ -[(e^{-f}(\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f)(s, x(s, \omega))] \int_0^t [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))ds + [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))] \]
\[ -[(e^{-f}(\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f)(s, x(s, \omega))]e^{f(t,x(t,\omega))} + (e^{-f}(\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f)(s, x(s, \omega))] \int_0^t [\left( \frac{\partial}{\partial s} + L_{s,\omega}\right)e^f](s, x(s, \omega))ds \]

First and last terms cancel each other as do the second and third.

3 implies 4.

**Limits of nonnegative martingales is a supermartingale.** Let \( X_n(t) \) be a sequence of non negative martingales with \( E[X_n(t)] = 1 \) and let \( X(t) = \lim_{n \to \infty} X_n(t) \) a.e. Then \( M(t) \) is a supermartingale.

**Proof.** Let \( E_k(s) = \{ \omega : \sup_n X_n(s) \leq k \} \). \( E_k(s) \in \mathcal{F}_s \) and \( E_k(s) \uparrow \Omega \)
\[ \int_{A \cap E_k(s)} X_n(s)dP = \int_{A \cap E_k(s)} X_n(t)dP \]

Let \( n \to \infty \) and use Fatou on the right and bounded convergence theorem on the left.
\[ \int_{A \cap E_k(s)} X(s)dP \geq \int_{A \cap E_k(s)} X(t)dP \]

Let \( k \to \infty \).
\[ \int_A X(s)dP \geq \int_A X(t)dP \]
or $E[X(t)|\mathcal{F}_s] \leq X(s)$ a.e.

The function $< \theta, x >$ is not bounded but can be approximated by smooth bounded functions and

$$\exp[< \theta, x(t) - x(0) > - \int_0^t < \theta, b(s, \omega) > ds - \frac{1}{2} \int_0^t < \theta, a(s, \omega) \theta > ds]$$

is a supermartingale.

$$E^P[\exp[< \theta, x(t) - x(0) > - \int_0^t < \theta, b(s, \omega) > ds - \frac{1}{2} \int_0^t < \theta, a(s, \omega) \theta > ds]] \leq 1$$

$$E^P[\exp[< \theta, x(t) >]] \leq \exp[t(c_1||\theta|| + c_2||\theta||^2)]$$

It is clear that $E[\exp[\lambda ||x(t)||]] < \infty$ for all $\lambda > 0$. The approximations can be constructed with uniform linear bounds.

Hence

$$X_{\theta}(t) = \exp[< \theta, x(t, \omega) - x(0, \omega) > - \frac{1}{2} \int_0^t < a(s, \omega) \theta, \theta > ds - \int_0^t < b(s, \omega), \theta > ds]$$

are martingales. If $y(t) = x(t) - x(0) - \int_0^t b(s, \omega) ds$, then

$$P[\sup_t ||y(t)|| \geq \ell] \leq c_1 \exp[-\frac{c_2 \ell^2}{t}]$$

4 implies 1.

Continue analytically. Replace $\theta$ by $i\theta$.

$$Y_{\theta}(t) = \exp[< i\theta, x(t, \omega) - x(0, \omega) > + \frac{1}{2} \int_0^t < a(s, \omega) \theta, \theta > ds + i \int_0^t < b(s, \omega), \theta > ds]$$

are martingales. Take

$$A(t) = \exp[-\frac{1}{2} \int_0^t < a(s, \omega) \theta, \theta > ds + i \int_0^t < b(s, \omega), \theta > ds]$$

Then $Y_{\theta}(t)A(t) - \int_0^T Y_{\theta}(s)dA(s)$ reduces to 1 with $f = e^{i<\theta,x>}$. Note that $y(t) - \int_0^t b(s, \omega) ds$ and $y_i(t)y_j(t) - \int_0^t a_{i,j}(s, \omega) ds$ are martingales.
Stochastic Integrals. Given \((\Omega, \mathcal{F}, x(s, \omega), P, \{a(s, \omega), b(s, \omega)\})\). A progressively measurable function \(e(s, \omega)\) with values in \(\mathbb{R}^d\) we want to define

\[
z(t, \omega) = \int_0^t < e(s, \omega), dx(s) > = \int_0^t < e(s, \omega), dy(s) > + \int_0^t < e(s, \omega), b(s, \omega) > ds
\]

It is only the \(dy\) integral that is a problem. Let us take for simplicity \(b = 0\). Take a subdivision \(t_j = \frac{j}{N}\)

**Step 1.** Assume \(e\) is uniformly bounded, is piecewise (in time) constant \(e = e_j(\omega)\) on \([t_{j-1}, t_j]\) which is \(\mathcal{F}_{t_{j-1}}\) measurable. Then for \(t_j \leq t \leq t_{j+1}\)

\[
z_N(t) = \sum_{1 \leq i \leq j} < e(t_{i-1}), y(t_i) - y(t_{i-1}) > + < e(t_j), y(t) - y(t_j) >
\]

\(z(\cdot)\) is linear in \(e\), almost surely continuous and for any such \(e\) it is a martingale and so is

\[
z^2(t) - \int_0^t < e(s, \omega), a(s, \omega)e(s, \omega) > ds
\]

and by Doob’s inequality

\[
E[\sup_{0 \leq s \leq T} |z(s)|^2] \leq 4E^P[\int_0^T < e(s, \omega), a(s, \omega)e(s, \omega) > ds]
\]

and

\[
\exp[z(t) - \int_0^t < e(s, \omega), b(s, \omega) > ds - \frac{1}{2} \int_0^t < e(s, \omega), a(s, \omega)e(s, \omega) > ds]
\]

are martingales.

**Step 2.** If \(e(s, \omega)\) is uniformly bounded and continuous we can approximate \(e(s, \omega)\) by \((e, \frac{\lfloor ns \rfloor}{n})\) which is again progressively measurable. We can pass to the limit. The limit exists and satisfy the same properties as before.

**Step 2.** Given a bounded progressively measurable \(e\) we define \(e_n\) for \(s \geq \frac{1}{n}\) by

\[
e_n(s) = n \int_{s-\frac{1}{n}}^s e(v, \omega) dv
\]

\[
\int_0^T \|e_n(s) - e(s)\|^2 ds \to 0 \text{ and therefore } z_n(t) \text{ has a limit.}
\]

**Step 3.** If \(E^P[\int_0^T \|e(s, \omega)\|^2 ds] < \infty\) we can truncate by

\[
e_\ell(s, \omega) = e(s, \omega) 1_{\|e(s, \omega)\| \leq \ell}
\]

and let \(\ell \to \infty\).
In conclusion we can define
\[ z(t) = \int_0^t < e(s, \omega), dx(s) > \]

provided
\[ E^P[ \int_0^T \| e(s, \omega) \|^2 ds ] < \infty \]

Then \( z(t) - \int_0^t < e(s, \omega, b(s, \omega)ds > \) is a square integrable martingale. If \( e \) is uniformly bounded then
\[ \exp[z(t) - \int_0^t < e(s, \omega, b(s, \omega)ds >] = \frac{1}{2} \int_0^t < e(s, \omega, a(s, \omega)e(s, \omega)ds > ds > \]

is martingale.

**The linear algebra of Stochastic Integrals.**

\[ [\Omega, \mathcal{F}_s, P, x(s, \omega), a(s, \omega), b(s, \omega)] \]

\( x \in \mathbb{R}^d, b \in \mathbb{R}^d, a \in S_d^+ S_d^+ \) is positive semidefinite \( d \times d \) matrices. Let \( y(t) = \int_0^t c(s, \omega)ds + \int_0^t e(s, \omega)dx(s) \) where \( c \in \mathbb{R}^n, e \in W_{n, d} \) where \( W_{n, d} \) is the set of \( n \times d \) matrices. Then \([\Omega, \mathcal{F}_s, P, y(s, \omega), \hat{a}(s, \omega), \hat{b}(s, \omega)] \) and \( y \in \mathbb{R}^n, \hat{b} \in \mathbb{R}^n, \hat{a} \in S_n^+, \hat{b} = c + eb \) and \( \hat{a} = eae^* \)

If \( X \) is Gaussian with mean \( \mu \) and covariance \( A \), \( Y = eX + c \) is Gaussian with mean \( e\mu + c \) and covariance \( eAe^* \).

**Itô’s Formula.** Let \( f(t, x) \) be a smooth bounded function. Let \( g_{\lambda, \theta}(t, x) = \lambda f(t, x) + < \theta, x > \).
\[ \exp[(g(t, x(t)) - g(0, x(0))) - \int_0^t H(s, \omega)ds] \]

is a martingale, where
\[ H(s, \omega) = \frac{\partial g}{\partial s}(s, x(s, \omega)) + \frac{1}{2} \sum_{i,j} a_{i,j}(s, \omega) \frac{\partial^2 g}{\partial x_i\partial x_j}(s, x(s, \omega)) + \sum_j b_j(s, \omega) \frac{\partial g}{\partial x_j}(s, x(s, \omega)) + \frac{1}{2} < a(s, \omega)(\nabla g)(s, \omega), (\nabla g)(s, \omega) > \]

Let \( y(t) = f(t, x(t)) - f(0, x(0)) \). Then
\[ [\Omega, \mathcal{F}_s, P, y(s, \omega), x(s, \omega), \hat{a}(s, \omega), \hat{b}(s, \omega)] \]

where
\[ \hat{b} = \frac{\partial f}{\partial s}(s, x(s, \omega)) + \frac{1}{2} \sum_{i,j} a_{i,j}(s, \omega) \frac{\partial^2 f}{\partial x_i\partial x_j}(s, x(s, \omega)) + \sum_j b_j(s, \omega) \frac{\partial f}{\partial x_j}(s, x(s, \omega)), b(s, \omega) \]
\[ \hat{a} = \begin{pmatrix} < a(s, \omega)(\nabla f)(s, x(s, \omega), (\nabla f)(s, \omega) > a(s, \omega)(\nabla f)(s, x(s, \omega)) \\ a(s, \omega)(\nabla f)(s, x(s, \omega)) a(s, \omega) \end{pmatrix} \]

Let us define a new process

\[ w(t) = f(t, x(t)) - f(0, x(0)) - \int_0^t f_s(s, x(s))ds - \int_0^t \langle (\nabla f)(s, x(s)), dx(s) \rangle \]

\[ dw = dy - f_s(s, x(s))ds - \langle (\nabla f)(s, x(s), dx(s)) \rangle \]

\[ [\Omega, \mathcal{F}_s, P, w(s, \omega), \tilde{a}(s, \omega), \tilde{b}(s, \omega)] \]

\[ \tilde{b} = \frac{1}{2} \sum_{i,j} a_{i,j}(s, \omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(s, x(s, \omega)) \]

\[ \tilde{a} = ((1, -(\nabla f)(s, x(s))), \tilde{a}(s, \omega)(1, -(\nabla f)(s, x(s)))) = 0 \]

\[ df(t, x(t)) = f_t dt + \langle (\nabla f), dx \rangle + \frac{1}{2} \sum_{i,j} a_{i,j}(t, \omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x(t, \omega)) dt \]