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Sequences, limits and series (continued).

If \( f(x) \) is function defined on \([a, \infty)\) i.e for all \( x \geq a \) we can talk about

\[
\lim_{x \to \infty} f(x) = \ell
\]

This means given any accuracy, i.e, a small number \( \epsilon \)

\[
|f(x) - \ell| \leq \epsilon
\]

for all sufficiently large \( x \), i.e. for \( x \geq A \). One can some times use L’Hospital’s rule to determine the limit. If \( f(x) \to \ell \) as \( \ell \to \infty \) and \( a_n = f(n) \), then \( a_n \to \ell \) as \( n \to \infty \). This is obvious because, if some thing is true for all \( x \geq A \), it is of course true for all integers larger than \( A \). The converse is not always true. \( f(x) = \sin 2x\pi \) is a periodic function and has no limit. But if you look at it points of the form \( y + 2n\pi \), then

\[
a_n = \sin(y + 2n\pi) = \sin y \to \sin y
\]

You can use this to calculate some limits.

Example:

\[
a_n = n^k e^{-cn}
\]

for any power \( k \) and \( c > 0 \). We look at function

\[
f(x) = x^k e^{-c x}
\]

for \( x \geq 1 \). We write it as

\[
f(x) = \frac{x^k}{e^{c x}}
\]

If \( k > 0, c > 0 \) this is of the form \( \frac{\infty}{\infty} \). If \( k \leq 0, c > 0 \) then the numerator is not big and the denominator gets big for large \( x \) and the limit is 0. When \( k > 0 \) apply L’Hospital’s rule. Then we reduce the problem to

\[
\frac{k}{c} \lim_{x \to \infty} \frac{x^{k-1}}{e^{c x}}
\]

i.e \( k \) become \( k - 1 \). The power reduces by 1. If we keep repeating the rule eventually \( k \) becomes 0 if we start from an integer value or less than 0 otherwise. In any case

\[
\lim_{x \to \infty} x^k e^{-c x} = 0
\]

and therefore

\[
\lim_{n \to \infty} n^k e^{-cn} = 0
\]
In other words $e^{cn}$ is much bigger than $n^k$ for any $k$. We write $e^{cn} >> n^k$.

Example: Ratio of polynomials.

$$a_n = \frac{a_0 n^k + a_1 n^{k-1} + \cdots + a_k}{b_0 n^p + b_1 n^{p-1} + \cdots + b_p}$$

with $a_0, b_0$ not equal to 0. Look at

$$f(x) = \frac{a_0 x^k + a_1 x^{k-1} + \cdots + a_k}{b_0 x^p + b_1 x^{p-1} + \cdots + b_p}$$

There are three cases. $k < p, k = p, k > p$. In the first case, after applying the rule $k$ times, we end up with

$$\frac{a_0 (k!)}{b_0 [p(p-1) \cdots (p-k+1)] x^{p-k} + \cdots}$$

which tends to 0. In the second case we end up with

$$\frac{a_0 (k!)}{b_0 (k!)} = \frac{a_0}{b_0}$$

which is the limit. In the third case we end up with

$$\frac{a_0 [k(k-1) \cdots (k-p+1)] x^{k-p} + \cdots}{b_0 (p!)}$$

which tends to $\pm \infty$ depending on the sign of $\frac{a_n}{b_0}$.

Example: If $f$ is continuous at $x = \ell$ and $a_n \rightarrow \ell$ then $f(a_n) \rightarrow f(\ell)$. This can be used; for instance

$$\lim_{n \rightarrow \infty} \sqrt{1 + e^{-n}} = 1$$

A sequence $a_n$ is bounded above if there is a number $M$ such that

$$a_n \leq M$$

for all $n$.

$$\lim_{n \rightarrow \infty} \sqrt{1 + e^{-n}} = 1$$

and a sequence $a_n$ is bounded below if there is a number $m$ such that

$$a_n \geq m$$

for all $n$. A bounded sequence is one that is bounded above and below. We can take the bounds to be $\pm M$ so that we have

$$|a_n| \leq M$$
for all $n$. A convergent sequence is always bounded, but the converse is not true. If a sequence is convergent to a limit $\ell$ then

$$|a_n - \ell| \leq 1$$

for $n \geq k$ some $k$. Then

$$M = \max\{|a_1|, \ldots, |a_k|, |\ell| + 1\}$$

will work. A monotone increasing sequence is bounded below and will converge to a limit if it is bounded above. The limit is actually the least upper bound, defined as the smallest number which is still an upper bound.

Example:

$$a_n = \arctan n$$

$$f(x) = \arctan x; \quad f'(x) = \frac{1}{1 + x^2} \geq 0$$

So $\arctan x$ is increasing. It is also bounded by $\frac{\pi}{2}$. In fact the limit is $\frac{\pi}{2}$.

**Home Work.**

Determine for each of the following sequences if it converges, is bounded but does not converge or unbounded.

1. $a_n = n \sin n$
2. $a_n = \sin(n^2 + 1)$
3. $a_n = \frac{n}{n+1}$
4. $a_n = (-2)^n$
5. $a_n = 11(-1)^n$