Integration of trigonometric polynomials.

To integrate
\[ \int \sin^m x \cos^n x \, dx \]

When one of them (either \( m \) or \( n \)) is odd. Let \( n = 2k + 1 \) be odd. Put \( u = \sin x \).
\[ \int u^m (1 - u^2)^k du \]

Let both be even. \( m = 2k \) and \( n = 2\ell \). Use \( \sin^2 x = \frac{1 - \cos 2x}{2} \) and \( \cos^2 x = \frac{1 + \cos 2x}{2} \)
\[ \frac{1}{2^{k+\ell}} \int (1 - \cos 2x)^k (1 + \cos 2x)^\ell \, dx \]

Simplify. Put \( y = 2x \). Various powers of \( \cos y \) show up. Odd powers can be handled. Even powers use half angle formula. Cut down the power by a factor 2. Repeat till done.

Example 1.
\[ \int \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x) \, dx \]
\[ = \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{x}{4} - \frac{1}{8} \int (1 + \cos 4x) \, dx \]
\[ = \frac{x}{4} - \frac{x}{8} - \frac{1}{32} \sin 4x + C = \frac{x}{8} - \frac{1}{32} \sin 4x + C \]

Example 2.
\[ \int e^{ax} \sin bx \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \]
\[ = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx \]
\[ (1 + \frac{a^2}{b^2}) \int e^{ax} \sin bx \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx \]
\[ \int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} [-b e^{ax} \cos bx + a e^{ax} \sin bx] + C \]

Example 3. Integrating
\[ \int \frac{dx}{a x^2 + bx + c} \]

Depends on the roots of the quadratic \( a x^2 + bx + c \). Two distinct real roots.
\[ a x^2 + bx + c = a(x - \alpha)(y - \beta) \]
Then partial fraction works. If we have two coincident roots, it is a square. Becomes

\[ \int \frac{dx}{a(x - \alpha)^2} \]

If no real roots it becomes by completing the square

\[ a[(x - \alpha)^2 + \beta^2] \]

Substituting \( x = \alpha + \beta y \) this reduces to

\[ \frac{1}{a\beta} \int \frac{dy}{1 + y^2} = \arctan y + C \]

Other things to remember from calculus I.

What is \( \sqrt{100} \)? 10.005

It is nearly 10. Say 10 + \( x \). Then

\[(10 + x)^2 = 100 + 20x + x^2 = 100.1\]

\[20x + x^2 = 0.1; \quad x \approx \frac{0.1}{20} = .005\]

For more accuracy try 10.005 + \( x \). Then

\[10.005^2 + (20.01)x + x^2 = 100.1\]

Ignore \( x^2 \). Next approximation is 10.005 + \( \frac{100.1 - (10.005^2)}{20.01} \) = 10.00499875

**Home work Problems:** Calculate the Integrals.

\[ \int_{0}^{1} x^{\frac{3}{2}} (1 - x)^2 \, dx \]
\[ \int_{0}^{\pi/2} (\sin x)^4 (\cos x)^4 \, dx \]
\[ \int_{0}^{1} xe^{2x} \, dx \]
\[ \int_{0}^{1} \frac{dx}{x^2 + 4} \]
\[ \int_{0}^{1} \frac{dx}{x^2 - 4} \]