A note on the numerical representation of surface dynamics in quasigeostrophic turbulence: Application to the nonlinear Eady model

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(6 August 2008)

Abstract

The quasigeostrophic equations consist of the advection of linearized potential vorticity coupled with advection of temperature at the bounding upper and lower surfaces. It is shown here that the surface temperature dynamics at scales below the deformation scale are poorly represented in models that use a finite-difference representation in the vertical. The result is that some aspects of surface dynamics in generic quasigeostrophic turbulence have been missed. We explore such effects in the context of the nonlinear Eady model, using a finite-difference representation of 3D quasigeostrophic flow in comparison with a model that explicitly advects temperature at the upper and lower surfaces. Theoretical predictions for the spectrum of turbulence in the nonlinear Eady model are reviewed and compared to the simulated flows. It is then shown that the finite difference method accurately represents the flow only down to a critical horizontal scale that decreases with decreasing vertical grid spacing. Specifically, in order to accurately represent dynamics at horizontal wavenumber $k$, the vertical grid spacing must satisfy $\delta_z \lesssim 3f/(Nk)$ (where $N$ is the buoyancy frequency and $f$ the Coriolis parameter).

1. Introduction

Synoptic-scale eddying flows in both the atmosphere and ocean lie in an asymptotic range well-described by the quasigeostrophic (QG) equations. Two phenomenological theories are typically invoked to describe the turbulent dynamics of such flows: geostrophic turbulence (GT, Charney, 1971) and surface quasigeostrophic turbulence (SQG, Blumen, 1978a; Held et al., 1995). These two theoretical ideas have been taken to describe complimentary but distinct flows, with the former assumed more generic and relevant to atmospheric and oceanic motions.

Geostrophic turbulence is based on the analogous structure of QG and two-dimensional dynamics, and leads to the prediction that total energy cascades to large scales, both horizontally and vertically, while potential enstrophy cascades to small scales. Barotropization and the upscale cascade are widely observed phenomena, and their existence has put GT on firm ground. GT, however, neglects boundaries entirely in the forward cascade — this is explicitly stated in Charney’s list of six limitations on the theory proposed in his 1971 paper. SQG, by contrast, considers

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only boundary dynamics: it is derived by assuming a semi-infinite volume with constant QGPV, bounded above or below by a surface on which temperature anomalies are advected by a geostrophic velocity field. The relationship between streamfunction and temperature is obtained by solving the homogeneous elliptic problem for the streamfunction in the interior, subject to boundary condition given by the temperature field on the surface. The conserved invariants, turbulent cascades and spectral slopes are different in SQG and GT: in SQG the total energy still undergoes an inverse cascade (albeit with a different spectral slope), but temperature variance (proportional to kinetic energy at the surface) cascades to small scales with a $-5/3$ spectral slope, while in GT, the kinetic energy spectrum has a $-3$ slope in the forward cascade range (Pierrehumbert et al., 1994).

With two boundaries bounding a region of constant QGPV, one obtains the system on which the linear baroclinic instability problem of Eady (1949) is based. The fully nonlinear system of this type was considered by Blumen 1978a. The two-layer Phillips model (Phillips, 1954) is simultaneously taken as a representation of the Eady model, for their baroclinic instability growth rates are very similar (see e.g. Pedlosky, 1987), and used as a model for geostrophic turbulence (since its forward cascade yields a kinetic energy spectrum slope that is close to $-3$; see, e.g. Larichev and Held 1995). The turbulent dynamics that arise in the Eady and Phillips model are not, however, the same, as argued by Blumen and shown explicitly by Hoyer and Sadourny (1982).

It is straightforward to see that the Eady model will behave differently than GT predicts at small scales: when the horizontal wavenumber of the motion $k$ is large enough that the corresponding vertical scale, $h = f/(kN)$ ($f$ is the Coriolis parameter and $N$ is the stratification), is small compared to the depth $H$ of the fluid, the surfaces temperature perturbations will not ‘feel’ the other boundary, and so the system will behave as a set of uncoupled SQG models, with $-5/3$ kinetic energy spectral slopes at each surface. In the Phillips model, by contrast, when the wavenumber $k$ is large enough that $f/(kN)$ is sufficiently less than the separation $H/2$ between the two layers, the vortex stretching term is suppressed and so the system becomes a set of decoupled layers, each controlled by 2D vorticity dynamics, resulting in a $-3$ kinetic energy spectral slope. The same result holds in the two-mode (barotropic-baroclinic) formulation (see e.g. Salmon, 1980): barotropic dynamics are always equivalent to 2D dynamics, and the baroclinic flow becomes dominated by vorticity dynamics when $k \gg NH/f$.

A standard finite difference QG model with only two vertical levels is isomorphic to the Phillips model (Pedlosky, 1987). Yet when the vertical resolution is increased, keeping the stratification and shear constant, the finite difference model should approach a representation of the Eady model. In this note we show explicitly that this is the case, but only up to a horizontal wavenumber that depends on the vertical resolution of the model. The wavenumber dependence can be understood as follows. Temperature $\theta$ in the quasigeostrophic approximation is related to the vertical derivative of the streamfunction $g\theta/\theta_0 = f\psi_z \approx f\delta\psi/\delta z$, where $\delta\psi/\delta z$ is the vertical grid spacing ($g$ is gravitational acceleration). Since vertical and horizontal scales are linked by the Prandtl ratio $N/f$, horizontal temperature signals will only be accurately represented at wavenumbers sufficiently smaller than $k\delta_z = f/(N \delta z)$. Surface effects, dominated by temperature advection, are therefore absent from low-vertical-resolution QG simulations, and in general, only partially represented down to scales of order $\ell_z = N\delta z/f$. An analogous argument was made by Solomon and Lindzen (2000), who demonstrated the necessity of sufficient resolution to model the barotropic instability of a point jet (see also McWilliams and Chow 1981, Fox-Rabinovitz and Lindzen 1993, and Snyder et al. 2003).

Why are the differences between the Eady and Phillips models not more widely known? This is probably due to the fact that at large scales, the models behave similarly, and so their subsequent accidental conflation has led to a neglect of their differences at smaller scales. Low vertical resolution in general, and the two-layer model in particular, are well-rationalized by the idea of
barotropization, which stems from Charney’s theory of geostrophic turbulence (Charney, 1971): if large-scale eddies are the result of an inverse cascade in both horizontal and vertical (scaled by $N/f$) dimensions, only the gravest few vertical modes should be necessary to represent those eddies (of course, similar logic should lead one to neglect higher horizontal wavenumbers as well). Moreover, the two-layer model has proved to be a tremendously useful tool for understanding much about atmospheric and oceanic eddy energy cycles and equilibration. In fact, one could say that so much has been understood through the lens of the two-layer model that those mesoscale phenomena that cannot be explained by it are often assumed to lie outside the realm of the quasigeostrophic approximation. A good example is the atmospheric energy spectrum (Nastrom and Gage, 1985), and in a companion paper to this one (Tulloch and Smith, 2008), we show that quasigeostrophic dynamics are adequate to explain those observations when surface effects are accounted for theoretically and resolved numerically.

2. Vertical finite-difference representations of the PV-streamfunction inversion

The quasigeostrophic equations driven by a mean zonal baroclinic wind, and bounded above and below by rigid surfaces consist of a conservation equation for the QG potential vorticity (PV) and two advection equations for the potential temperature at each surface (equations stated in numerous other works — see Vallis 2006 or Tulloch and Smith 2008, for example). The potential vorticity $q$ and potential temperature $\theta$ are related to the streamfunction $\psi$ by the elliptic problem

$$q = \nabla^2 \psi + \left( \frac{f^2 N^2}{\partial z^2} \psi \right)_z, \quad \text{and} \quad \theta|_{z=0,H} = \psi|_{z=0,H}. \quad (2.1)$$

A typical direct vertical discretization of this relation is (assuming constant $N^2$ and grid spacing $\delta_z$)

$$\Gamma_{nm}\psi_m = \frac{f^2}{N^2 \delta_z^2} \begin{cases} \psi_2 - \psi_1, & n = 1 \\ \psi_{n-1} - 2\psi_n + \psi_{n+1}, & n = 2 \ldots N - 1 \\ \psi_{N-1} - \psi_N, & n = N \end{cases} \quad (2.2)$$

where $N$ is the total number of layers (which should not be confused with $N$, the buoyancy frequency). Bretherton (1966) showed that, instead of considering non-constant $\psi_z$ at the upper and lower boundaries, one can equivalently consider constant potential temperature at the boundaries with $\delta$-sheets of potential vorticity just inside the boundaries, given by

$$q(z) = -\frac{f^2}{N^2 \delta_z} \left. \psi_z \right|_{z=H} \delta(z - H) + \frac{f^2}{N^2 \delta_z} \left. \psi_z \right|_{z=0} \delta(z).$$

The discrete operator $\Gamma_{nm}$ effectively includes an approximation of the $\delta$-sheets that is accurate to $O(N\delta_z/f)$ (see appendix of Smith, 2007), so surface temperature dynamics at horizontal scales smaller than $O(N\delta_z/f)$ are not captured.

3. The structure of nonlinear Eady turbulence

Here we consider the nonlinear Eady problem (Eady, 1949): $f$-plane, uniform stratification and shear, and the fluid is bounded above and below by rigid surfaces, depth $H$ apart. The mean interior PV gradient $Q_y = 0$ and the mean surface temperature gradients are equal $\Theta_y(H) = \Theta_y(0)$, thus $q = 0$ and the motion is determined solely by temperature advection on the boundaries. Assuming horizontal periodicity and taking the Fourier transform of the remaining equations, one has

$$\hat{\theta} + \hat{J}(\hat{\psi}, \hat{\theta}) + ik(U\hat{\theta} + \hat{\psi}\Theta_y) = 0, \quad z = 0, H \quad (3.3)$$
with hatted variables denoting spectral components (e.g. $\hat{\psi}(x, y, z, t) = \sum_K \exp(iK \cdot \mathbf{x})\hat{\psi}_K(z, t)$, with $\mathbf{K} = (k, \ell)$ the horizontal wavenumber vector), the hatted Jacobian denoting a double sum over wavenumbers of the spectral products, and the temperature-streamfunction relation given by $K = KNH/f$ with $K = |\mathbf{K}|$. We refer to this representation of Eady dynamics as the Blumen model, following Blumen (1978b).

One can understand the turbulent dynamics of the Eady model by considering the advection equations at each surface in the limits of large and small scales, separately. At the upper boundary ($z = H$) the streamfunction is related to the surface temperatures as

$$
\hat{\psi}(H, t) = \frac{H}{\mu \sinh \mu} \left[ \cosh \left( \mu \frac{z}{H} \right) \hat{\theta}(H, t) - \cosh \left( \mu \frac{z-H}{H} \right) \hat{\theta}(0, t) \right].
$$

(3.4)

The variable $\mu = KNH/f$ is the nondimensional wavenumber and $K = |\mathbf{K}|$. We refer to this representation of Eady dynamics as the Blumen model, following Blumen (1978b).

At large scales ($\mu \ll 1$), both $\sinh \mu$ and $\tanh \mu$ are approximated by $\mu$, so that $\hat{\psi}(H, t) \simeq (H/\mu^2)\hat{\theta}(H, t) - \hat{\theta}(0, t) \equiv -(H/\mu^2)\Delta \hat{\theta}$. A similar relation exists that arises at the bottom boundary, giving $\psi(0, t) \simeq \hat{\psi}(H, t)$. Subtracting the upper and lower advection equations, one has

$$
(\Delta \hat{\theta})_t + \hat{J}(\hat{\psi}, \Delta \hat{\theta}) + ik(U \Delta \hat{\theta} + \hat{\psi} \Theta_y) \simeq 0
$$

and so the equation for the temperature difference between the two surfaces is isomorphic to 2D vorticity flow (since $\Delta \hat{\theta} = -\mu^2 \hat{\psi}/H$ in this limit). At small scales ($\mu \gg 1$), on the other hand, $\sinh \mu \to \infty$ but $\tanh \mu \sim 1$, so that $\hat{\psi}(H, t) \simeq (H/\mu)\hat{\theta}(H, t)$, and similarly at the bottom, so that each surface obeys SQG dynamics, independent of the other surface.

In between these scale limits, where $\mu \sim 1$, baroclinic instability pumps energy into the eddying flow. Thus the small-scale limit is governed by the direct cascade, while the large-scale limit is controlled by the inverse cascade. Surface potential and kinetic energies in the inverse cascade of baroclinic turbulence and in the SQG direct cascade are all expected to obey a $-5/3$ slope, thus in non-linear Eady turbulence there should be no spectral break in the surface energy spectra, and a $-5/3$ surface spectrum should dominate all scales. On the other hand, there should be a horizontal scale dependence in the interior flow. At large scale, temperature signals reach through the full depth of the domain, yielding a quasi-barotropic flow, and so the interior spectrum should also approach a $-5/3$ slope. At small scales, the temperature signals are trapped near their respective surfaces, and so the interior flow should be quiescent.

4. Numerical simulations

We present a series of simulations, using two discretizations with the Eady mean state: (i) using full vertical resolution and the finite difference operator (2.2), and (ii) using the “Blumen model”, which advects only the upper and lower temperature fields. The calculations are performed in $K$-space horizontally, with wavenumber 1 just filling the domain. The nonlinear terms are calculated using a de-aliased fast Fourier transform. Forward cascades are dissipated using a scale-selective exponential cutoff filter which acts explicitly on $K \geq 2K_{\text{max}}/3$ but has little effect until the highest wavenumbers. The horizontal resolution of the simulations is $K_{\text{max}} = 511$, or $1024^2$ resolution in grid space. To preserve the dynamics and spectral slopes, the slow inverse cascades are not dissipated in this series of simulations.

Fig. 1 shows the results of a series of simulations, using the standard QG formulation with the discretization in (2.2). The kinetic energy density $K^2|\hat{\psi}|^2$ is plotted versus $K$ in the top level
for simulations with an increasing vertical resolution: $N = 4, 8, 16, 32,$ and 64 levels. All of the simulations have the same deformation scale ($K_d = fL/NH = 5$), and seed energy ($E = 10^{-3}$) at $K_0 = 10$ which grows due to Eady baroclinic instability, leading to a dual cascade. The peak linear growth rate is near $1.6K_d \approx 8$ and there is no linear baroclinic growth at wavenumbers above $2.4K_d \approx 12$. Since there is no large scale drag to halt the cascade and equilibrate the motion, we show a partial time average (from $t = 4.5$ to $t = 5$ in nondimensional time) of the KE density, normalized by $\epsilon^{2/3}$, where $\epsilon = U f^2 / H N^2 (\bar{\theta}|_{z=H} - \bar{\theta}|_{z=0})$ is the baroclinic energy generation rate, in order to compare the different vertical resolutions. For the two-layer case (not shown), the spectra approach $-3$ slopes at large wavenumbers. However, as the vertical resolution of the vertically discretized simulations increases, the spectra approach a $-5/3$ slope up to a wavenumber that increases with vertical resolution.

Plotted in the inset of figure 1 are “roll-off” wavenumbers $K_{\text{rolloff}}$ (defined to be where the spectral slope of the KE spectrum drops to $K^{-7/3}$) against $K_{\delta z} = f/(N\delta z)$ for each $\delta z = H/N$. The dependence of $K_{\text{rolloff}}$ on $K_{\delta z}$ is roughly linear with a best fit slope of 0.34 (indicated by the dashed line), indicating that in order to resolve a wavenumber $K_{\text{max}}$, a vertical grid scale spacing $\delta z \leq 0.34f/(NK_{\text{max}})$ is required.

The kinetic energy spectra at depths throughout the flow for both the 64-level simulation (dashed lines) and the Blumen model simulation (solid lines) are shown in Fig. 2. The Blumen simulation is normalized and averaged in the same way as the vertically discrete interior QG solution, and only three of its levels are plotted. The 64 level simulation is clearly a good representation of the nonlinear Eady model at this horizontal resolution; at higher horizontal resolutions, however, the spectrum will fail to resolve smaller horizontal scales unless its vertical resolution is concomitantly increased.

The implied resolution requirements suggested here are similar to but different than those suggested by Barnier et al. (1991). Barnier et al. argued that it is necessary to resolve horizontally the smallest baroclinic deformation scale. By contrast, the results here put the onus on the vertical resolution: whatever the horizontal resolution, the vertical resolution must be sufficiently fine in order for those horizontally included scales to accurately represent surface temperature dynamics. Still, both arguments, as well as those of papers cited in the introduction, imply the need for consistent horizontal and vertical resolutions, and all suggest their quotient should follow the Prandtl ratio, $N/f$.

5. Summary and discussion

We have demonstrated that, with sufficient vertical resolution, the numerical turbulent solution to the nonlinear Eady model with standard vertical finite differencing converges to the ‘exact’ solution, computed using a formulation proposed originally by Blumen (1978a) that explicitly advects the surface temperatures. This convergence results from the fact that, in the standard vertical discretization, surface boundary conditions are effectively implemented as a finite approximation to the PV delta-sheets of Bretherton (1966). The error to that approximation leads to a horizontal dependence on the vertical resolution: horizontal length scales of order the Prandtl ratio times the vertical grid spacing and smaller are not properly represented in the standard formulation.

In Tulloch and Smith (2008) we propose a generalized remedy to this problem that does not require excessive vertical resolution. The proposed formulation decomposes the the QG streamfunction into a sum of interior and surface components that solve the associated elliptic inversion problems. Truncating the interior modes to a computationally manageable set yields an efficient and accurate representation of the full system at all depths. The formulation also allows for the generation of simplified models, including both well-known approximations and new ones.
In the ocean, recent simulations by Klein et al. (2008) indicate that the kinetic energy spectrum exhibits a shallowing near the surface, consistent with SQG effects near the surface. Scott and Wang (2005) observed the advective flux of surface kinetic energy to be in the inverse direction, and Capet et al. (2008) point out that the surface advective flux in SQG is also in the inverse direction. While it does not seem likely that balanced surface dynamics can explain the entire picture, it is likely that it plays a part. In a forthcoming paper, we explore the degree to which the a model that properly represents both the interior and surface dynamics can explain the ocean surface dynamics at the sub-mesoscales.

Acknowledgments.

We acknowledge helpful conversations with Glenn Flierl and Kevin Hamilton. This work was supported by NSF OCE-0620874.
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Figure 1: Kinetic energy density versus horizontal wavenumber magnitude $K$ in the top layer of a series of Eady-forced QG simulations (with $nz = 4, 8, 16, 32, 64$ layers, $\beta = 0, U_z = 1$ and deformation wavenumber $k_d = fL/NH = 5$. Since there is no large-scale dissipation, the spectra shown are normalized by the baroclinic generation rate for each and then averaged in time between $t = 4.5$ and $t = 5$ for each simulation. The inset shows measured roll-off wavenumbers $K_{\text{rolloff}}$ (where the spectral slope is $K^{-7/3}$) versus the prediction $K_{\delta z} = f/(N\delta z)$. The best fit line is $K_{\text{rolloff}} = 0.34K_{\delta z} + 8$. 
Figure 2: Comparison of kinetic energy density in the Blumen model versus the standard QG formulation with 64 layers. The dashed gray lines are KE density at the mid-depths of layers $z_1, z_2, z_4, z_8, z_{16}$ and $z_{32}$ in the layered QG model, while the solid black lines are KE density at $z = 0, z = z_1$ and $z = z_{32}$ in the Blumen model.