# Geometry of equivariant compactifications of $\mathbb{G}_a^n$

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## 1 Introduction

In this paper we begin a systematic study of *equivariant* compactifications of $\mathbb{G}_a^n$. The question of classifying non-equivariant compactifications was raised by F. Hirzebruch ([10]) and has attracted considerable attention since (see [5], [16], [13] and the references therein). While there are classification results for surfaces and non-singular threefolds with small Picard groups, the general perception is that a complete classification is out of reach.

On the other hand, there is a rich theory of equivariant compactifications of reductive groups. The classification of normal equivariant compactifications of reductive groups is combinatorial. Essentially, the whole geometry of the compactification can be understood in terms of (colored) fans. In particular, these varieties do not admit moduli. For more details see [15], [4], [2] and the references therein.

Our goal is to understand *equivariant* compactifications of $\mathbb{G}_a^n$. The first step in our approach is to classify possible $\mathbb{G}_a^n$-structures on simple varieties, like projective spaces or Hirzebruch surfaces. Then we realize general smooth $\mathbb{G}_a^n$-varieties as appropriate (i.e., equivariant) blow-ups of simple varieties. This gives us a geometric description of the moduli space of equivariant compactifications of $\mathbb{G}_a^n$.

In section 2 we discuss general properties of equivariant compactifications of $\mathbb{G}_a^n$ ($\mathbb{G}_a^n$-varieties). In section 3 we classify all possible $\mathbb{G}_a^n$-structures on projective spaces $\mathbb{P}^n$. In section 4 we study curves, paying particular attention to non-normal examples. In section 5 we carry out our program completely for surfaces. In particular, we classify all possible $\mathbb{G}_a^2$-structures on minimal rational surfaces. In section 6 we turn to threefolds. We give a classification of $\mathbb{G}_a^3$-structures on smooth projective threefolds with Picard group of rank 1. Each section ends with a list of examples and open questions.

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# 2 Generalities

We work in the category of algebraic varieties over $F = \overline{\mathbb{Q}}$.

## 2.1 Definitions

**Definition 2.1** *Let $\mathbb{G}$ be a connected linear algebraic group. An algebraic variety $X$ admits a (left) $\mathbb{G}$-action if there exists a morphism $\varphi : \mathbb{G} \times X \to X$, satisfying the standard compatibility conditions. A $\mathbb{G}$-variety $X$ is a variety with a* fixed *(left) $\mathbb{G}$-action such that the stabilizer of a generic point is trivial and the orbit of a generic point is dense.*

For example, a normal $\mathbb{G}_m^n$-variety is a toric variety.

**Definition 2.2** *A morphism of $\mathbb{G}$-varieties is a morphism of algebraic varieties commuting with the $\mathbb{G}$-action. A $\mathbb{G}$*-isomorphism *is an isomorphism in the category of $\mathbb{G}$-varieties. A $\mathbb{G}$-equivalence is a diagram*

$$\begin{array}{ccc} \mathbb{G} \times X_1 & \stackrel{(\alpha, j)}{\longrightarrow} & \mathbb{G} \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \stackrel{j}{\longrightarrow} & X_2 \end{array}$$

*where $\alpha \in \mathrm{Aut}(\mathbb{G})$ and $j$ is an isomorphism (in the category of algebraic varieties).*

Clearly, every $\mathbb{G}$-isomorphism is a $\mathbb{G}$-equivalence. We shall omit $\mathbb{G}$ if the group is understood.

One of our main observations in this paper is that classification of simple $\mathbb{G}$-varieties up to equivalence, even projective spaces, is a non-trivial problem. This is in marked contrast to the situation for toric varieties. A toric variety admits a unique structure as a $\mathbb{G}_m^n$-variety (up to equivalence). Here is a sketch of the argument for a projective toric variety $X$: Consider the connected component of the identity $\mathrm{Aut}(X)^0$ of the automorphism group $\mathrm{Aut}(X)$. This is an algebraic group which acts trivially on the Picard group of $X$. We pick a very ample line bundle and consider $\mathrm{Aut}(X)^0$ as a closed subgroup of the corresponding group $\mathrm{PGL}_N$. In particular, $\mathrm{Aut}(X)^0$ is a linear algebraic group. The key ingredient now is the statement that all maximal tori in $\mathrm{Aut}(X)^0$ are conjugate (cf. 11.4 in [1]). Evidently, the maximal torus acts faithfully on $X$, and the action has a dense open orbit. This proves the claim.

Every $\mathbb{G}$-variety contains an open subset isomorphic to $\mathbb{G}$. We denote by $D$ the complement of this open subset and call $D$ the *boundary*. If $X$ is normal, Hartog's theorem implies that $D$ must be a divisor (the complement to $D$ is affine). Otherwise, we normalize and observe that the normalization is an isomorphism over $\mathbb{G}$.

## 2.2 Line bundles and linearizations

**Proposition 2.3** *Let $X$ be proper and normal algebraic variety. Then the action of $\mathbb{G}_a^n$ on the Picard group* $\mathrm{Pic}(X)$ *is trivial and every line bundle on $X$ admits a unique linearization, up to scalar multiplication.*

*Proof.* Uniqueness follows from Prop. 1.4 p. 33 in [14]. (The only relevant hypothesis is that $X$ is geometrically reduced, which is part of our assumptions.) The proof of Prop. 1.5 p. 34 implies that every line bundle on $X$ has a linearization. (Here we use that $X$ is proper and that the Picard group of the group $\mathbb{G}$ is trivial. To get the fact that the action of $\mathbb{G}$ on the $\mathrm{Pic}(X)$ is trivial we need the normality of $X$.)

**Corollary 2.4** *Retain the assumptions of Proposition 2.3. Consider a base-point free linear series $W \subset H^0(X, L)$, stable under the action of $\mathbb{G}_a^n$. Then the induced map $f : X \to \mathbb{P}(W^*)$ (here we use the* geometric *convention) is $\mathbb{G}_a^n$-equivariant.*

**Theorem 2.5** *Let $X$ be a proper and normal $\mathbb{G}_a^n$-variety. Then* $\mathrm{Pic}(X)$ *is freely generated by the classes of the irreducible components $D_j$ ($j = 1, ..., t$) of the boundary divisor $D$. The cone of effective Cartier divisors $\Lambda_{\mathrm{eff}}(X) \in \mathrm{Pic}(X)_{\mathbb{R}}$ is given by*

$$\Lambda_{\mathrm{eff}}(X) = \oplus_{j=1}^{t} \mathbb{R}_{\geq 0}[D_j].$$

*Proof.* Choose an effective divisor $A \subset X$ and consider the representation of $\mathbb{G}_a^n$ on the projectivization $H^0(X, \mathcal{O}(A))$ (here we use that $X$ is normal). This representation has a fixed point corresponding to an effective divisor supported at the boundary. To show that there are no relations between the classes $[D_1], ..., [D_t]$ it suffices to observe that there exist no functions without zeros and poles on $\mathbb{G}_a^n$.

**Remark 2.6** *Every effective cycle on a $\mathbb{G}$-variety is rationally equivalent to a cycle supported on the boundary. (Here we are using the fact the $\mathbb{G}$ is affine.)*

## 2.3 Vector fields and the anticanonical line bundle

**Theorem 2.7** *Let $X$ be a smooth and proper $\mathbb{G}$-variety. Then the anticanonical class is a sum of classes of the irreducible components of the boundary $D$ with coefficients which are all $\geq 1$. If $\mathbb{G} = \mathbb{G}_a^n$ then the coefficients are all $\geq 2$.*

*Proof.* We first introduce some general terminology and exact sequences. Let $Y$ be a smooth variety and $B$ a smooth divisor. Let $\mathcal{T}_Y \langle -B \rangle$ denote the sheaf whose sections are vector fields with logarithmic zeros along $B$. If $x_1, \ldots, x_n$ are local coordinates for $Y$ so that $B$ is given by $x_1 = 0$, then local sections of $\mathcal{T}_Y \langle -B \rangle$ take the form $x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$. There are two natural exact sequences

$$\begin{array}{rcl} 0 & \to & \mathcal{T}_Y \langle -B \rangle \to \mathcal{T}_Y \to \mathcal{N}_{B/Y} \to 0 \\ 0 & \to & \mathcal{T}_Y(-B) \to \mathcal{T}_Y \langle -B \rangle \to \mathcal{T}_B \to 0 \end{array}$$

where $\mathcal{N}_{B/Y}$ is the normal bundle to $B$.

Let $v \in H^0(Y, \mathcal{T}_Y)$ be a vector field. By definition, $v$ *vanishes normally to order one along* $B$ if its image in $H^0(B, \mathcal{N}_{B/Y})$ is zero. If $v$ arises from the action of a one-parameter group stabilizing $B$ then it vanishes normally to order one along $B$. If $v$ vanishes normally to order one along $B$, we can consider the corresonding element $w \in H^0(B, \mathcal{T}_B)$. If $w = 0$ then we say that $v$ *vanishes to order one*. If $v$ arises from a one-parameter group fixing $B$ then it vanishes to order one. Generally, if $v$ vanishes to order $N-1$ (resp. normally to order $N$) then we can consider the corresponding element $w \in H^0(B, \mathcal{N}_{B/Y}(-(N-1)B))$ (resp. $H^0(B, \mathcal{T}_B(-(N-1)B))$). If $w = 0$

then we say that $v$ vanishes normally (resp. vanishes) to order $N$ along $B$; in particular, $v$ may be regarded as a section of $H^0(Y, \mathcal{T}_Y \langle -B \rangle (-(N-1)B))$ (resp. $H^0(Y, \mathcal{T}_Y(-NB))$).

We now prove the theorem. First assume $X$ is a $\mathbb{G}$-variety and $D_i$ an irreducible component of its boundary. We take $Y \subset X$ as the complement to the singular locus of $D_i$ and $B = Y \cap D_i$. Let $\{v_1, \ldots, v_n\} \in H^0(X, \mathcal{T}_X)$ be invariant vector fields spanning the Lie algebra of $\mathbb{G}$, and $M_j$ (resp. $N_j$) the order of vanishing (resp. normal vanishing) of $v_j$ along $B$. By definition, $N_j = M_j$ or $M_j + 1$. Consider the exterior power $\sigma = v_1 \wedge v_2 \wedge \ldots \wedge v_n \in H^0(X, \Lambda^n \mathcal{T}_X)$; we bound (from below) the order of vanishing $a_i$ of $\sigma$ along $D_i$. Evidently $a_i \geq M_1 + \ldots + M_n$ but in fact slightly more is true. Using the adjunction isomorphism

$$\Lambda^n \mathcal{T}_Y | B = \Lambda^{n-1} \mathcal{T}_B \otimes \mathcal{N}_{B/Y}$$

we obtain

$$a_i \geq \min_{j=1,\ldots,n} (M_1 + \ldots + M_{j-1} + N_j + M_{j+1} + \ldots + M_n).$$

For instance, since each $N_j > 0$ we obtain that $a_i > 0$. It follows that the canonical divisor is linearly equivalent to $-\sum_i a_i D_i$, where each $a_i > 0$.

Now assume that $X$ is a $\mathbb{G}_a^n$-variety. Let $v_1$ be a vector field arising from the action of a one-parameter subgroup that fixes $D_i$, so that $v_1$ vanishes to order one and $M_1 \geq 1$. We claim $v_1$ necessarily vanishes normally to order *two* i.e. $N_1 \geq 2$. Consider the resulting element $w \in H^0(B, \mathcal{N}_{B/Y}(-B)) = \mathrm{End}(\mathcal{N}_{B/Y})$, which exponentiates to give the induced $\mathbb{G}_a^1$ action on the normal bundle. Since $\mathbb{G}_a^1$ has no characters, this action is trivial and $w = 0$. The previous inequality guarantees that $a_i \geq 2$. Hence the canonical divisor is linearly equivalent to $-\sum_i a_i D_i$ where each $a_i > 1$.

**Remark 2.8** *The proof only uses the fact that $X$ is smooth at generic points of the boundary divisor.*

**Corollary 2.9** *Let $X$ be a smooth projective $\mathbb{G}_a^n$-variety with irreducible boundary $D$, which is fixed under the action. Then $X = \mathbb{P}^n$.*

*Proof.* We see that $M_j \geq 1$ for each $j$. It follows that $K_X = -rD$ where $r \geq n+1$, i.e. $X$ is Fano of index $r \geq n+1$. Hence $X = \mathbb{P}^n$ and $r = n+1$.

**Corollary 2.10** *Let $X$ be a smooth projective $\mathbb{G}_a^n$-variety with irreducible boundary $D$. Assume that the subgroup of $\mathbb{G}_a^n$ fixing $D$ has dimension $n-1$. Then $X = \mathbb{P}^n$ or $X = Q_n$, the quadric hypersurface of dimension $n$.*

*Proof.* After reordering, we obtain that $M_j \geq 1$ for $j = 1, \ldots, n-1$. It follows from the proof of Theorem 2.7 that $N_j \geq 2$ for $j = 1, \ldots, n-1$ and $N_n \geq 1$. In particular, $r \geq n$, i.e., $X$ is Fano of index $r \geq n$. Hence $X = \mathbb{P}^n$ or $Q_n$.

## 2.4 A dictionary

Let $L(\mathbb{G}_a^n)$ and $U(\mathbb{G}_a^n)$ denote the Lie algebra and the enveloping algebra of $\mathbb{G}_a^n$. Since $\mathbb{G}_a^n$ is commutative $U(\mathbb{G}_a^n)$ is isomorphic to a polynomial ring in $n$ variables. Let $R = U(\mathbb{G}_a^n)/I$ and assume that $\mathrm{Spec}(R)$ is supported at the origin; we use $\ell(R)$ and $m_R$ to denote the length and the maximal ideal of $R$. Since $I$ contains all the homogeneous polynomials of sufficiently large degree $d$, the elements of $L(\mathbb{G}_a^n)$ act (via the regular representation) as nilpotent matrices on $R$. We can exponentiate to obtain an algebraic representation

$$\rho : \mathbb{G}_a^n \to \mathrm{Aut}_F(R)$$

of dimension $\ell(R)$.

For concreteness, we introduce some additional notation. We consider $L(\mathbb{G}_a^n)$ as a vector space over $F$ with a distinguished basis $S_j = \frac{\partial}{\partial x_j}$ so that $U(\mathbb{G}_a^n) = F[S_1, \ldots, S_n]$. The representation $\rho$ is obtained by multiplying by $\exp(x_1 S_1 + \ldots + x_n S_n) \in R$.

**Proposition 2.11** *Choose an $F$-basis $\{\mu_1, \mu_2, \ldots, \mu_{\ell(R)}\}$ for $R$. Then the coordinate functions $f_1, f_2, \ldots, f_{\ell(R)}$ arising from the expansion*

$$\exp(x_1 S_1 + \ldots + x_n S_n) = \sum_{j=1}^{\ell(R)} f_j(x_1, \ldots, x_n)\mu_j$$

*form a basis for the solution space $V$ of the system of partial differential equations*

$$g[f(x_1, \ldots, x_n)] = 0 \text{ for each } g \in I \subset F[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}].$$

*In particular,* $\dim V = \ell(R)$.

*Proof.* It suffices to exhibit a basis with the desired properties. This basis is constructed with Gröbner basis techniques [3] chapter 15. Consider homogeneous lexicographic order, which induces a total order on the set of all monomials in $S_1, \ldots, S_n$. Let $\mathrm{Init}(I)$ be the initial ideal for $I$ and $\{g_i\}$ a Gröbner basis for $I$, i.e., the initial terms of the $g_i$ generate $\mathrm{Init}(I)$. The monomials $\mu_1, \mu_2, \ldots, \mu_{\ell(R)}$ not contained in $\mathrm{Init}(I)$ form a basis for $R$ (cf. proof of [3] Theorem 15.17.) Each monomial $\mu$ admits a unique representation in $R$

$$\sum_{j=1}^{\ell(R)} c_j(\mu)\mu_j.$$

The division algorithm (with respect to our Gröbner basis) implies that $c_j(\mu) = 0$ whenever $\mu$ strictly precedes $\mu_j$ in the total order.

The formula

$$\frac{\partial}{\partial x_i} \exp(x_1 S_1 + \ldots + x_n S_n) = S_i \exp(x_1 S_1 + \ldots + x_n S_n)$$

implies that the $f_j$ are solutions to our system of partial differential equations.

We put a total order on the monomials in the $x_i$: $x_1^{a(1)} \ldots x_n^{a(n)}$ precedes $x_1^{b(1)} \ldots x_n^{b(n)}$ whenever $S_1^{b(1)} \ldots S_n^{b(n)}$ precedes $S_1^{a(1)} \ldots S_n^{a(n)}$. We claim each $f_j$ contains a term proportional to $x_1^{m(1)} \ldots x_n^{m(n)}$, where $\mu_j = S_1^{m(1)} \ldots S_n^{m(n)}$, and this is the initial term of $f_j$ with respect to our order. Since the $f_j$ have distinct initial terms, they are linearly independent.

To prove the claim, note that $\exp(x_1 S_1 + \ldots + x_n S_n)$ expands as a sum of nonzero terms

$$C_{a(1),\ldots,a(n)} x_1^{a(1)} \ldots x_n^{a(n)} S_1^{a(1)} \ldots S_n^{a(n)}.$$

Each term yields an element in $R$ of the form

$$C_{a(1),\ldots,a(n)} x_1^{a(1)} \ldots x_n^{a(n)} \sum_{j=1}^{\ell(R)} c_j(S_1^{a(1)} \ldots S_n^{a(n)})\mu_j,$$

where $c_j = 0$ whenever $S_1^{a(1)} \ldots S_n^{a(n)}$ precedes $\mu_j$. In particular, the surviving nonzero terms

$$x_1^{a(1)} \ldots x_n^{a(n)} S_1^{m(1)} \ldots S_n^{m(n)}$$

all have the property that $x_1^{a(1)} \dots x_n^{a(n)}$ precedes $x_1^{m(1)} \dots x_n^{m(n)}$ in the order. Furthermore, the unique term

$$C_{m(1),\dots,m(n)} x_1^{m(1)} \dots x_n^{m(n)} S_1^{m(1)} \dots S_n^{m(n)}$$

corresponds to the initial term of $f_j$.

To complete the proof, we show that

$$V := \{f(x_1, \dots, x_n) : g[f] = 0 \text{ for each } g \in I\}$$

has dimension $\ell(R)$. Since $I$ contains all the polynomials of degree $d$, each solution is polynomial with total degree $< d$. There is a natural pairing between $F[x_1, \dots, x_n]$ and $F[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$

$$\langle g, f \rangle = g[f]|_{(0,\dots,0)}.$$

This induces a perfect pairing between homogeneous polynomials and operators of a given degree. Note that $V = \{f : \langle g, f \rangle = 0 \text{ for each } g \in I\}$ which implies that $\dim V = \ell(R)$.

We collect some basic properties of $\rho$:

**Proposition 2.12** *The $\mathbb{G}_a^n$-representations $\rho$ and $V$ are dual, and $\rho$ has a nondegenerate orbit (i.e., a cyclic vector) in $R$. The representation $\rho$ is faithful iff* $\mathrm{Spec}(R) \subset \mathrm{Spec}(F[S_1, \dots, S_n])$ *is nondegenerate.*

*Proof.* The vector space $V$ has a natural $\mathbb{G}_a^n$-action by translations. The first statement is clear from the preceding discussion. Consider the orbit of $1 \in R$

$$\rho(x_1, \dots, x_n) \cdot 1 = \sum_{i=1}^{\ell(R)} f_j \mu_j.$$

The linear independence of the $f_j$ implies that this orbit is nondegenerate in $R$. The final statement is clear; indeed, $S_i$ acts trivially iff $\mathrm{Spec}(R) \subset \{S_i = 0\}$.

**Remark 2.13** *Note that $V = \rho^*$ has a natural structure as an $R$-module of length $\ell(R)$. Indeed, it coincides with the dualizing module $\omega_R$. This can be seen using Macaulay's method of inverse systems (see [3] chapter 21.2). It follows that $\rho$ is self-dual iff $R$ is Gorenstein.*

A translation invariant subspace $V \subset F[x_1, \ldots, x_n]$ of dimension $\ell$ corresponds to a representation $\rho : \mathbb{G}_a^n \to \mathrm{GL}_\ell$ with a fixed cyclic vector $v$. Indeed, we may regard the elements of $V$ as coordinate functions on the nondegenerate orbit $\rho(\mathbb{G}_a^n) \cdot v$. Consider the ideal $I$ of constant coefficient differential operators annihilating $V$ and write

$$R = F[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]/I.$$

Since $I$ contains all monomials of sufficiently large degree, $\mathrm{Spec}(R)$ is supported at the origin. The pairing introduced in the proof of Proposition 2.11 may be used to show that $\mathrm{Spec}(R)$ has length $\ell$.

We summarize our results in the following dictionary:

**Theorem 2.14** *There is a one-to-one correspondence among the following:*

1. *subschemes* $\mathrm{Spec}(R) \subset \mathrm{Spec}(F[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}])$ *supported at the origin of length* $\ell = \ell(R)$*;*
2. *translation invariant subspaces* $V \subset F[x_1, \ldots, x_n]$ *of dimension* $\ell$*;*
3. *isomorphism classes of pairs* $(\rho, v)$ *such that* $\rho : \mathbb{G}_a^n \to \mathrm{GL}_\ell$ *is a representation and* $v$ *is a cyclic vector (i.e.,* $\rho(\mathbb{G}_a^n) \cdot v$ *is nondegenerate).*

We now turn to a case of particular interest. Assume that $S_1, \ldots, S_n$ form a basis for the maximal ideal $m_R$, so that $\ell(R) = n + 1$. Then the corresponding representation $\rho_R$ is faithful and the induced action on $\mathbb{P}(R)$ has a *dense* orbit. Hence for any Artinian local $F$-algebra, exponentiating the action of $m_R$ on $R$ yields a $\mathbb{G}_a^{\ell(R)-1}$-structure on $\mathbb{P}(R) = \mathbb{P}^{\ell(R)-1}$. Conversely, assume we are given a $\mathbb{G}_a^{\ell-1}$-structure on $\mathbb{P}^{\ell-1}$. By Proposition 2.3, this action admits a unique linearization on the line bundle $\mathcal{O}(+1)$, and we obtain a faithful representation $\rho : \mathbb{G}_a^{\ell-1} \to \mathrm{GL}_\ell$. The corresponding ring of differential operators $R$ is Artinian local of length $\ell$. Since $\rho$ is faithful, $S_1, \ldots, S_{\ell-1}$ form a basis for $m_R$.

We summarize this discussion in the proposition[1]:

**Proposition 2.15** *The following are equivalent:*

1. *Artinian local* $F$*-algebras* $R$ *of length* $\ell = \ell(R)$*, up to isomorphism;*
2. *equivalence classes of* $\mathbb{G}_a^{\ell-1}$*-structures on* $\mathbb{P}^{\ell-1}$*.*

[1] see also Prop. 5.1 of [12]

### 2.5 Examples and questions

1. Not every point in the boundary $D$ is contained in the closure of a 1-parameter subgroup. Construction: Blow up $\mathbb{P}^2$ in a point at the boundary. Blow up again a point in the exceptional divisor. Every 1-parameter subgroup in $\mathbb{P}^2$ is a line.

2. The theorem 2.7 fails for non-equivariant compactifications of $\mathbb{G}_a^n$. For example, let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a hypersurface of bidegree $(1, d)$ with $d \geq 4$. Then the anticanonical class $\mathcal{O}(2, 3-d)$ is not contained in the interior of the effective cone.

3. Suppose $X$ is a smooth projective $\mathbb{G}_a^n$-variety with finitely many $\mathbb{G}_a^n$-orbits. Is $X$ rigid as an algebraic variety?

## 3 Projective spaces

In this section we study $\mathbb{G}_a^n$-structures on projective spaces. Notice that every $\mathbb{P}^n$ has a distinguished structure as a $\mathbb{G}_a^n$-variety. The translation action on the affine space $\mathbb{A}^n$ extends to an action on $\mathbb{P}^n$, fixing the hyperplane at infinity. We denote this action by $\tau_n$. It corresponds to the Artinian ring $F[S_1, ..., S_n]/[S_j S_j,\ i, j = 1, ..., n]$. It is easy to see that *every* $\mathbb{G}_a^n$-structure on $\mathbb{P}^n$ admits a specialization to $\tau_n$.

In the following propositions, we classify $\mathbb{G}_a^n$-structures on projective spaces of small dimension, *up to equivalence* (cf. 2.2). The first natural invariant is the Hilbert-Samuel function of the corresponding Artinian ring $R$, defined by $\chi_R(k) = \ell(m_R^k / m_R^{k+1})$.

**Proposition 3.1** *There is a unique $\mathbb{G}_1^a$-structure on $\mathbb{P}^1$.*

*Proof.* It is a consequence of Proposition 2.15.

**Proposition 3.2** *There are two distinct $\mathbb{G}_a^2$-structures on $\mathbb{P}^2$. They are given by the following representations of $\mathbb{G}_a^2$:*

$$\tau_2(a_1, a_2) = \begin{pmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$

*and*

$$\rho(a_1, a_2) = \begin{pmatrix} 1 & a_1 & a_2 + \frac{1}{2}a_1^2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

*They correspond to the quotients of $F[S_1, S_2]$ by the ideals $I_1 = [S_1S_2, S_2^2, S_1^2]$ and $I_2 = [S_1S_2, S_2 - S_1^2]$.*

*Proof.* It suffices to classify Artinian local $F$-algebras $R$ of length three up to isomorphism. If the tangent space has dimension two then $R = F[S_1, S_2]/I_1$. If the tangent space to $R$ has dimension one then $R = F[S_1, S_2]/I_2$ (it is clear that this is the only one). These correspond to the representations $\tau_2$ and $\rho$ respectively.

**Proposition 3.3** *There are four distinct $\mathbb{G}_a^3$-structures on $\mathbb{P}^3$. They correspond to the quotients of $F[S_1, S_2, S_3]$ by the following ideals:*

$$\begin{aligned} I_1 &= [S_1^2 - S_2, S_1S_2 - S_3, S_1S_3] \\ I_2 &= [S_1^2 - S_2, S_1S_2, S_1S_3] \\ I_3 &= [S_1^2, S_1S_2 - S_3, S_2^2] \\ I_4 &= [S_1^2, S_1S_2, S_2^2, S_2S_3, S_3^2, S_1S_3]. \end{aligned}$$

*Proof.* If the tangent space to $R$ has dimension one or three, then $R$ is necessarily the quotient of the polynomial ring by $I_1$ or $I_3$. If the tangent space has dimension two then $\ell(m_R^2/m_R^3) = 1$. Consider the nonzero symmetric quadratic form

$$q : m_R/m_R^2 \times m_R/m_R^2 \to m_R^2/m_R^3.$$

Let $S_3$ generate $m_R^2$. Then there exist $S_1, S_2 \in m_R$ spanning $m_R/m_R^2$ such that the quadratic form equals $S_1S_2$ or $S_1^2$.

**Proposition 3.4** *There are ten distinct $\mathbb{G}_a^4$-structures on $\mathbb{P}^4$. They corre-*

*spond to the quotients of $F[S_1, S_2, S_3, S_4]$ by the following ideals:*

$$\begin{aligned}
I_1 &= [S_1^2 - S_2, S_1S_2 - S_3, S_1S_3 - S_4, S_2S_3, S_1S_4] \\
I_2 &= [S_1^2 - S_3, S_1S_2, S_2^2, S_1S_3 - S_4, S_1S_4] \\
I_3 &= [S_1^2 - S_3, S_1S_2, S_2^2 - S_3, S_1S_3 - S_4, S_1S_4] \\
I_4 &= [S_1^2 - S_3, S_1S_2 - S_3, S_2^2, S_1S_3 - S_4, S_1S_4] \\
I_5 &= [S_1^2 - S_3, S_1S_2 - S_4, S_2^2, S_1S_3, S_1S_4, S_2S_3] \\
I_6 &= [S_1^2 - S_3, S_2^2 - S_4, S_1S_2, S_1S_3, S_2S_4] \\
I_7 &= [S_1^2 - S_4, S_2S_3 - S_4, S_1S_2, S_1S_3, S_2^2, S_3^2, S_1S_4] \\
I_8 &= [S_1^2, S_2^2, S_3^2, S_1S_2 - S_4, S_1S_3, S_2S_3] \\
I_9 &= [S_1^2 - S_4, S_2^2, S_3^2, S_1S_2, S_1S_3, S_1S_4, S_2S_3] \\
I_{10} &= [S_iS_j,\ i, j = 1, ..., 4]
\end{aligned}$$

*Proof.* We consider the possible shapes of the Hilbert-Samuel function $\chi_R$. In the cases where $\chi_R(1) = 1$ or 4 it is clear that the only possibilities are $I_1$ and $I_{10}$ respectively.

Assume that $\chi_R(1) = 2$, $\chi_R(2) = 1$ and $\chi_R(3) = 1$. Choose $S_4 \in m_R^3, S_3 \in m_R^2$, and $S_1, S_2 \in m_R$ which span the maximal ideal of the graded ring associated to $R$. For a suitable choice of $S_2$ and $S_4$, we may assume that $S_2S_3 = 0$ and $S_1S_3 = S_4$. Consider the map $s_2 : m_R/m_R^2 \to m_R^2/m_R^3$ induced by multiplying by $S_2$. If $s_2 = 0$ then we may choose $S_3$ so that $S_1^2 = S_3$; we obtain $I_2$. If $s_2 \neq 0$ and $S_2 \notin \ker(s_2)$ then we may choose $S_1$ so that $S_1S_2 = 0$. However, $S_4 \neq 0$ implies $S_1^2 \neq 0$, so after rescaling $S_1, S_2$, and $S_4$ we obtain $I_3$. If $s_2 \neq 0$ and $S_2 \in \ker(s_2)$ then (after rescaling $S_3$) we obtain $S_1S_2 = S_3$. Again, after rescaling $S_1$ and $S_2$, $S_1^2 \neq 0$ and we obtain $I_4$.

Assume that $\chi_R(1) = 2$ and $\chi_R(2) = 2$. Then we choose $S_1, ..., S_4$ such that $S_3$ and $S_4$ are in $m_R^2$ and $S_1, S_2$ are independent modulo $m_R^2$. The ring structure on $R$ is determined by the vector-valued quadratic form

$$q : m_R/m_R^2 \times m_R/m_R^2 \to m_R^2/m_R^3.$$

This corresponds to choosing a codimension 1 subspace of $\mathrm{Sym}^2(m_R/m_R^2)$. Up to changes of coordinates in $S_1$ and $S_2$, each such subspace is spanned by vectors $\{S_1^2, S_2^2\}$ and $\{S_1S_2, S_1^2\}$. This gives the cases $I_5$ and $I_6$.

Assume that $\chi_R(1) = 3$ and $\chi_R(2) = 1$. This corresponds to $I_7, I_8$, and $I_9$. The ring structure is determined by the quadratic form $q$ (with values in $F$), which has rank $3, 2$ or 1.

**Proposition 3.5** *There are finitely many distinct $\mathbb{G}_a^5$-structures on $\mathbb{P}^5$.*

*Proof.* These are written out explicitly in Suprunenko [17] pp. 136-150.

The above discussion mirrors the classification of algebras of commutative nilpotent matrices in the book [17]. (Notice a misprint in the classification of algebras corresponding to $\mathbb{G}_a^3$-structures on $\mathbb{P}^3$ on page 134). The arguments in this book yield a classification of Artinian algebras of length $n+1$ with the following Hilbert-Samuel functions (though the author does not make this explicit):

| $\chi_R(0)$ | $\chi_R(1)$ | $\chi_R(2)$ | $\chi_R(3)$ | ... | $\chi_R(n)$ |
|---|---|---|---|---|---|
| 1 | $n$ | 0 | 0 | ... | 0 |
| 1 | 1 | 1 | 1 | ... | 1 |
| 1 | 2 | 1 | 1 | ... | 0 |
| 1 | 3 | 1 | 1 | ... | 0 |
| 1 | 2 | 2 | 1 | ... | 0 |

In particular, there are finitely many algebras with each of these Hilbert-Samuel functions $\chi_R$. This suffices to obtain a complete classification of $\mathbb{G}_a^n$ structures on $\mathbb{P}^n$ for $n \leq 5$.

Beginning with dimension 6 we obtain moduli. As an example, let us consider Artinian rings with Hilbert-Samuel function of the shape $(1, n-k, k, 0, ..., 0)$ (for suitable $k$). These correspond to $k$-dimensional spaces of quadratic forms in $n-k$ variables (up to coordinate transformations of the $n-k$ variables). The quadratic forms are obtained by dualizing the natural map

$$\mathrm{Sym}^2(m_R/m_R^2) \to m_R^2/m_R^3.$$

For example, if $n = 6$ and $k = 2$ the moduli space is birational to the moduli space of elliptic curves.

**Example 3.6** *There exists at least one* 1*-parameter family of inequivalent $\mathbb{G}_a^6$-structures on $\mathbb{P}^6$.*

If $n = 8$ and $k = 3$ we get the moduli space of genus 5 curves. If $n = 9$ and $k = 3$ we obtain the moduli space of K3 surfaces of degree 8. The appearance of these K3 surfaces and the genus 5 curves is quite interesting. Can it be explained geometrically, in terms of birational maps between different $\mathbb{G}_a^n$-structures?

For a good general introduction to Artinian rings (and many further references) see [11].

**Proposition 3.7** *The projective space $\mathbb{P}^n$ admits a unique $\mathbb{G}_a^n$-structure with finitely many orbits. It corresponds to the Artinian ring $F[S_1, ..., S_n]/I$, where*

$$\begin{aligned} I &= [S_1^2 - S_2, S_1S_2 - S_3, ..., S_1S_{n-1} - S_n, S_iS_j,\ i+j > n] \\ &\simeq [S_1^i - S_i, S_iS_j,\ i+j > n]. \end{aligned}$$

*Proof.* There is a unique fixed point under the $\mathbb{G}_a^n$-action. Projecting from it gives a $\mathbb{P}^{n-1}$ with finitely many $\mathbb{G}_a^{n-1}$-orbits. By the inductive hypothesis, this $\mathbb{P}^{n-1}$ has the indicated structure. The Artinian ring $R(\mathbb{P}^{n-1})$ for $\mathbb{P}^{n-1}$ is a quotient of the Artinian ring $R(\mathbb{P}^n)$. Let $S_n \in R(\mathbb{P}^n)$ be a non-zero element mapped to $0 \in R(\mathbb{P}^{n-1})$. Since $R(\mathbb{P}^{n-1}) \simeq F[T_1]/[T_1^n]$, there exists an element $S_1 \in R(\mathbb{P}^{n-1})$ such that $S_1^{n-1} \neq 0$. Then $S_1^n = cS_n$ for some constant $c \in F$. If $c = 0$ the action has infinitely many orbits. Otherwise, (after rescaling) we obtain the desired ideal.

## 3.1 Examples and questions

1. Untwisting different actions on $\mathbb{P}^2$: Take $\mathbb{P}^2$ with the $\rho$-action and choose a generic one-parameter subgroup. Let $C$ be the conic obtained as the closure of a generic orbit of the one-parameter subgroup. The curve $C$ is tangent to the line at infinity at the fixed point $p$. Blow up the fixed point *on* $C$ 3 times. Contract the strict transforms of the line at infinity, and the first two exceptional curves.

2. In general, there does not exist an *irreducible* variety parametrizing all $\mathbb{G}_a^n$-structures on $\mathbb{P}^n$ (fails for $n = 7$). This is related to the fact that a general length 8 subscheme of $\mathbb{A}^4$ is not a limit of 8 distinct points (this was pointed out to us by Iarrobino, see [11]). For example, consider the subschemes cut out by 7 general quadrics in 4 homogeneous variables. These subschemes deform only to subschemes of the same type. Clearly, not every Artinian local $F$-algebra of length 8 has tangent space of dimension $\geq 4$ (the curvilinear ones have 1-dimensional tangent space).

3. Give an explicit factorization for $\mathbb{G}_a^n$-equivariant birational automorphisms of the projective space $\mathbb{P}^n$.

4. Give a dictionary between $\mathbb{G}_a^n$-structures on smooth quadrics $Q_n$ and certain Artinian rings (with additional structure).

## 4 Curves

**Proposition 4.1** *Every* smooth *proper* $\mathbb{G}_a^1$*-variety is isomorphic to* $\mathbb{P}^1$ *with the standard translation action* $\tau_1$*.*

*Proof.* Exercise.

**Lemma 4.2** *Let* $V$ *be the standard representation of* $\mathbb{G}_a^1$*. Then* $\mathrm{Sym}^n(V)$ *has a filtration*

$$0 \subset \mathrm{F}_0 \subset \mathrm{F}_1 \subset ... \subset \mathrm{F}_n = \mathrm{Sym}^n(V)$$

*which is compatible with the* $\mathbb{G}_a^1$*-action and such that* $\mathrm{F}_i \simeq \mathrm{Sym}^i(V)$ *and* $\mathrm{F}_{i+1}/\mathrm{F}_i$ *is the 1-dimensional trivial representation. Furthermore, every stable subspace of* $\mathrm{Sym}^n(V)$ *arises in this way.*

*Proof.* Exercise.

**Remark 4.3** *The Jordan canonical form gives us a complete description of representations of* $\mathbb{G}_a^1$*. They are isomorphic to direct sums of the representations* $\mathrm{Sym}^n(V)$*.*

**Proposition 4.4** *Every proper* $\mathbb{G}_a^1$*-variety* $C$ with an equivariant projective embedding *is isomorphic to* $\mathbb{P}^1$ *embedded by a complete linear series with the translation action.*

*Proof.* Clearly, the normalization of the curve $C$ is isomorphic to $\mathbb{P}^1$. Furthermore, the normalization map $\nu : \mathbb{P}^1 \to C$ is equivariant and an isomorphism away from the fixed point $P_\infty \in \mathbb{P}^1$. The morphism $\nu$ is given by some basepoint free linear series $W$ on $\mathbb{P}^1$, which is stable under the $\mathbb{G}_a^1$-action. In particular, $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(n)) \simeq \mathrm{Sym}^n(V)$ (where $V$ is the standard representation). By the previous Lemma, each stable proper subspace of $\mathrm{Sym}^n(V)$ corresponds to a linear series on $\mathbb{P}^1$ *with basepoints.* This concludes the proof.

Our next goal is to classify proper 1-dimensional $\mathbb{G}_a^1$-varieties $C$. Clearly, the normalization $\tilde{C}$ has to be isomorphic to $\mathbb{P}^1$ with the standard action $\varphi$ and with the conductor-ideal vanishing at the fixed point $t = 0$. Hence it suffices to classify conductor-ideals $I \subset F[t]|_{t=0}$, stable under the group action.

**Theorem 4.5** *The only conductor-ideals* $I$ *which are stable under the group action* $\varphi$ *are preimages of some semigroup* $\Sigma \subset \mathbb{Z}_{<0}$ *under the valuation homomorphism.*

*Proof.* Consider the complete local ring $F[[t]]$ with maximal ideal m. We make the identification

$$\frac{\mathrm{m}}{\mathrm{m}^{n+2}} \simeq \mathrm{Sym}^n(V)$$

(it follows from the definition of the action $\varphi : t \mapsto t \cdot (1 - at + a^2t^2 - ...)$ extended to the completion). By lemma 4.2, all subspaces stable under the action of $\mathbb{G}_a$ coincide with preimages of subsets of the valuation group.

## 4.1 Examples and questions

1. Let $C \in \mathbb{P}^2$ be a cuspidal cubic plane curve. Then it is a proper $\mathbb{G}_a^1$-variety which does not admit $\mathbb{G}_a^1$-equivariant projective embeddings. Indeed, the action on the normalization $\mathbb{P}^1 = \tilde{C}$ of $C$ is given by

$$\varphi : t \mapsto \frac{t}{1+at}.$$

   Note that the underlying topological space of $C$ is just $\mathbb{P}^1$ and that all local rings coincide, *except* at the cusp 0. The local ring $\mathcal{O}_{C,0}$ is equal to the ideal generated by $t^2, t^3$ in the ring $F[t]|_{t=0}$. This ideal is fixed under the action of $\varphi$. Therefore, the action descends to $C$. By (4.4), $C$ does not admit a $\mathbb{G}_a^1$-equivariant projective embedding.

2. Describe versal deformation spaces of non-normal proper $\mathbb{G}_a^1$-varieties together with the $\mathbb{G}_a^1$-action on these spaces.

# 5 Surfaces

Throughout this section $X$ will be a smooth proper $\mathbb{G}_a^2$-variety.

**Proposition 5.1** *Let $E \subset X$ be a $(-1)$-curve. Then there exists a morphism of $\mathbb{G}_a^2$-varieties $X \to X'$ which blows down $E$.*

*Proof.* This follows from proposition 2.3 and corollary 2.4.

**Proposition 5.2** *Every $\mathbb{G}_a^2$-surface $X$ admits a $\mathbb{G}_a^2$-equivariant morphism onto $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$.*

*Proof.* This follows from the existence of minimal models for rational surfaces.

## 5.1 Hirzebruch surfaces

Let $X$ be a $\mathbb{G}_a^2$-variety. Assume that $X$ is isomorphic to $\mathbb{F}_n$ as an algebraic variety with $n > 0$. Its zero-section $e$ is stabilized under the group action, as is the distinguished fiber $f$. Let $\xi_n$ be the line bundle on $X$ corresponding to the section at infinity. There is an induced equivariant morphism

$$\mu : X \to \mathbb{P}(H^0(X, \xi_n)^*) = \mathbb{P}^{n+1}.$$

The image $\mu(X)$ is the cone over a smooth rational normal curve of degree $n$; $\mu$ contracts $e$ to the vertex of this cone.

We compute the representation of $\mathbb{G}_a^2$ on $H^0(X, \xi_n)^*$. It has a distinguished one-dimensional fixed subspace $W_1$ corresponding to the vertex. The resulting representation on $H^0(X, \xi_n)^*/W_1$ can be easily understood geometrically. It has a one dimensional kernel and the corresponding faithful representation

$$\mathbb{G}_a^1 \to \mathrm{GL}(H^0(X, \xi_n)^*/W_1)$$

is the $n$-fold symmetric power of the standard two-dimensional representation. Here we are using the fact that $\mu(X)$ is the cone over a rational normal curve of degree $n$.

Choose a basis $S_1, S_2 \in L(\mathbb{G}_a^2)$ such that $S_1$ acts nontrivially and $S_2$ acts trivially as matrices on $H^0(X, \xi_n)^*/W_1$. As a matrix on $H^0(X, \xi_n)^*$, $S_2$ has image contained in $W_1$, and $S_1 S_2 = 0$. We have already seen that $S_1^n \neq 0$ as a matrix on $H^0(X, \xi_n)^*/W_1$.

We consider two possible cases: either $S_1^{n+1} \neq 0$ or $S_1^{n+1} = 0$. In the first case, we apply the following fact about nilpotent matrices.

**Lemma 5.3** *Let $S_1$ be an $(n+2) \times (n+2)$ nilpotent matrix such that $S_1^{n+1} \neq 0$. Then the centralizer of $S_1$ consists of the algebra of matrices generated by $S_1$ and the identity.*

*Proof.* This follows from a straightforward induction once we put $S_1$ in Jordan canonical form.

The lemma implies that $S_2$ may be written as some polynomial of $S_1$. The fact that the images of $S_2$ and $S_1^{n+1}$ both lie in $W_1$ implies that $T = cS^{n+1}$ for some $c \neq 0$. In this case we have

$$H^0(X, \xi_n)^* = \rho_R \text{ where } R = F[S_1, S_2]/[S_1 S_2, S_2 - S_1^{n+1}].$$

Furthermore, $S_2$ acts nontrivially on the distinguished fiber $f$.

We now assume that $S_1^{n+1} = 0$. In this case we have

$$H^0(X, \xi_n)^* = \rho_R \text{ where } R = F[S_1, S_2]/[S_1S_2, S_1^{n+1}],$$

and the action is trivial along the distinguished fiber.

In conclusion:

**Proposition 5.4** *Let $X$ be a $\mathbb{G}_a^2$-variety as above and let $\xi_n$ denote the line bundle corresponding to the section at infinity. If the action on the distinguished fiber is nontrivial then the representation*

$$\mathbb{G}_a^2 \to \mathrm{GL}(H^0(X, \xi_n)^*) = \mathrm{GL}_{n+2}$$

*is equivalent to* $\exp(a_1S_1 + a_{n+1}S_2)$ *where* $S_2 = S_1^{n+1} \neq 0$ *and* $S_1S_2 = 0$. *If the distinguished fiber is fixed under the action then the representation is equivalent to* $\exp(a_1S_1 + a_{n+1}S_2)$ *where* $S_1^{n+1} = 0$ *and* $S_1S_2 = 0$.

A geometrical interpretation is obtained as follows. Let $W$ be the $(n+2)$-dimensional representation of $\mathbb{G}_a^2$ described above. Then each $X$ admits an equivariant birational morphism into $\mathbb{P}(W)$ and corresponds to the closure of some nondegenerate orbit. To classify the surfaces $X$ it suffices to classify the nondegenerate $\mathbb{G}_a^2$ orbits of $\mathbb{P}(W)$ modulo automorphisms, i.e. the $\mathbb{G}_a^2$-automorphisms of $\mathbb{P}(W)$.

In the first case, these are exactly the automorphisms commuting with the action of $S_1$, i.e., the homotheties and the matrices

$$\exp(a_1S_1 + a_2S_1^2 + \ldots + a_{n+1}S_1^{n+1}).$$

Note that this gives $\mathbb{P}(W)$ the structure of a $\mathbb{G}_a^{n+1}$-variety, which has a dense open orbit equal to the complement of the distinguished hyperplane. In particular, any two nondegenerate orbit closures in $\mathbb{P}(W)$ are related by an automorphism of $\mathbb{P}(W)$. It follows that $X$ is unique up to equivalence.

In the second case, these automorphisms include the homotheties and the matrices

$$\exp(a_1S_1 + a_2S_1^2 + \ldots + a_nS_1^n + a_{n+1}S_2).$$

Again, $\mathbb{P}(W)$ has the structure of a $\mathbb{G}_a^{n+1}$-variety, and any two nondegenerate orbit closures are related by an automorphism. It follows that $X$ is unique up to equivalence.

These arguments yield the following:

**Proposition 5.5** *The Hirzebruch surfaces $\mathbb{F}_n$ with $n > 0$ each have two distinct $\mathbb{G}_a^2$-structures, including a unique structure with a nontrivial action on the distinguished fiber. The second structure is obtained by taking an elementary transformation of the structure on $\mathbb{F}_{n-1}$. The product $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ has a unique $\mathbb{G}_a^2$-structure, induced from the $\mathbb{G}_a^1$ actions on each factor.*

The elementary transformation involves blowing-up the intersection of the zero section and the distinguished fiber of $\mathbb{F}_{n-1}$, and then blowing-down the proper transform of this fiber. To prove the last statement, we project from a fixed point of $\mathbb{F}_0$. The image is $\mathbb{P}^2$ with $\mathbb{G}_a^2$ acting by translation.

## 5.2 Examples and questions

1. Interesting *singular* surfaces admitting $\mathbb{G}_a^2$-structures: Del Pezzo surface of degree 5 with an isolated $A_4$-singularity.
2. Can the $\mathbb{G}_a^2$-structures on a given (smooth) surface have moduli?
3. Classify $\mathbb{G}_a^2$-structures on projective surfaces with log-terminal singularities and Picard number 1.

# 6 Threefolds

**Theorem 6.1** *Let $X$ be a smooth projective $\mathbb{G}_a^3$-variety such that the boundary $D$ is irreducible. Then $X$ is one of the following:*

1. *$X = \mathbb{P}^3$, $D$ a hyperplane (the possible $\mathbb{G}_a^3$-structures were listed in 3.3);*
2. *$Q_3 \subset \mathbb{P}^4$ is a smooth quadric, $D$ a tangent hyperplane section. It has a unique $\mathbb{G}_a^3$-structure (described in the proof).*

*Proof.* We know that $-K_X = r \cdot D$ where $r \geq 2$ (2.7). Therefore, $X$ is a Fano variety of index $r \geq 2$ and it is rational. Furthermore, if it has index $= 2$ then the subgroup fixing $D$ has dimension one (by Corollary2.10). We first consider the case where the index $> 2$.

We show there is a unique $\mathbb{G}_a^3$-structure on $Q_3$, and that the boundary $D$ is necessarily equal to a tangent hyperplane section. First let us convince ourselves that a quadric with a tangent hyperplane is equivariant. Consider $\mathbb{P}^3$ with the translation action $\tau_3$, which fixes the hyperplane at infinity $P$.

Blow up a smooth conic curve $C \subset P \subset \mathbb{P}^3$ and blow down the proper transform of $P$. Now we prove that there are no other $\mathbb{G}_a^3$-structures on $Q_3$. Any $\mathbb{G}_a^3$-action on $Q_3$ has a fixed point $p$. Projecting from $p$ gives an equivariant birational map $f : Q_3 \dashrightarrow \mathbb{P}^3$. The induced map from the blow-up of $Q_3$ at $p$ to $\mathbb{P}^3$ is the blow-up of a conic contained in the proper transform of $D$ (where $D$ is the boundary of $\mathbb{G}_a^3$ in $Q_3$). The proper transform of $D$ is a plane $P$. The classification of $\mathbb{G}_a^3$-structures on $\mathbb{P}^3$ implies that $P$ is *fixed* under the action of $\mathbb{G}_a^3$ on $\mathbb{P}^3$ (cf. 3.3).

We return to the case where index equals two. By Furushima's classification of non-equivariant compactifications of $\mathbb{G}_a^3$ (cf. [5], [6]), $X$ is a codimension 3 linear section of the Grassmannian $\mathrm{Gr}(2,5)$. Consider the action induced on $F$, the variety of lines on $X$. Any line on $X$ has normal bundle equal to $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(+1)$ and there is always a line of the second type (cf. [8]). Choose such a line $L$ stable under the $\mathbb{G}_a^3$ action (we are choosing a fixed point on the locus of lines of the second type in $F$). Projecting from $L$ gives an equivariant birational map

$$\pi_L : X \dashrightarrow Q_3$$

(cf. pp. 112 of [8]).

By [6], there are two cases to consider. In the first case the boundary $D \subset X$ is non-normal, with singular locus $L$. The total transform of $D$ consists of a hyperplane section $H \subset Q_3$. The image of the boundary $D$ is a smooth rational curve of degree 3, contained in $H$. We have already shown that there is a unique $\mathbb{G}_a^3$-structure on $Q_3$, which does not admit any smooth rational curves of degree 3 contained in the boundary and stable under the action (the only stable curve in $H$ is the distinguished ruling).

In the second case the boundary $D \subset X$ is a normal singular Del Pezzo surface of degree 5 with an isolated $A_4$-singularity. The curve $L \subset D$ is the unique $(-1)$-curve in the minimal resolution of $D$. Under $\pi_L$, $D$ is mapped birationally (and equivariantly!) to a tangent hyperplane section $H \subset Q_3$. The subgroup fixing $H$ has dimension 2, so the same holds for $D$. This contradicts Corollary 2.10.

## 6.1 Examples and questions

1. A singular projective $\mathbb{G}_a^3$-variety with one irreducible boundary divisor on which the $\mathbb{G}_a^3$-action is trivial. Construction: Blow up a pair

of intersecting lines in $\mathbb{P}^3$. Then blow down the proper transform of the plane containing them. The resulting variety is a singular quadric hypersurface in $\mathbb{P}^4$.

2. $\mathbb{G}_a^3$-equivariant flop: Consider a quadric hypersurface $Q^* \subset \mathbb{P}^4$ with an isolated singularity $p$. Let $Y$ be the blow up of $Q^*$ at $p$. Let $E \equiv \mathbb{P}^1 \times \mathbb{P}^1$ be the exceptional divisor. Blowing down $E$ in different directions yields smooth threefolds $X_1$ and $X_2$. It suffices to exhibit a $\mathbb{G}_a^3$-structure on $Q^*$ (all the constructions are natural and equivariant). This structure is obtained by using the fact that $Q^*$ is a cone over $\mathbb{F}_0$.

3. $\mathbb{G}_a^3$-equivariant flip: Consider the cone $V$ over the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$. There exists a $\mathbb{G}_a^3$-structure on $V$ with a non-singular fixed point $p$. Indeed, given $\mathbb{P}^2$ with the translation action the cone $V$ has a unique $\mathbb{G}_a^3$-structure with fixed boundary divisor. Let $Y$ be the blow-up of $V$ with center in $p$ and let $Z$ be the variety obtained from $Y$ by contracting the proper transform of the ruling through $p$. In particular, $Z$ is obtained from $Y$ as a small contraction and $K_Z$ is not $\mathbb{Q}$-Cartier. The variety $Z$ is isomorphic to the cone over a cubic scroll $\mathbb{F}_1 \subset \mathbb{P}^4$. The flipped threefold $X$ is a small resolution of $Z$.

4. If $X$ is a smooth projective $\mathbb{G}_a^3$-variety and the action on the boundary is trivial is $X = \mathbb{P}^3$?

# References


[1] A. Borel, *Linear Algebraic Groups,* W.A. Benjamin, New York, (1969).

[2] M. Brion, *Sur la géométrie des variétés sphériques,* Comment. Math. Helv. **66 (2)**, 237–262, (1991).

[3] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry,* Springer-Verlag (1995).

[4] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton NJ, (1993).

[5] M. Furushima, *The complete classification of compactifications of* $\mathbf{C}^3$ *which are projective manifolds with second Betti number one*, Math. Ann. **297 (4)**, 627–662, (1993).

[6] M. Furushima, *The structure of compactifications of* $\mathbb{C}^3$, Proc. of Japan Acad., **68**, Ser. A, 33–36, (1992).

[7] M. Furushima, *Non-projective compactifications of* $\mathbb{C}^3$. *(I)* Kyushu J. Math. **50 (1)**, 221–239, (1996).

[8] M. Furushima, N. Nakayama, *The Family of Lines on the Fano Threefold* $V_5$, Nagoya Math. J. **116**, 111–122 (1989).

[9] M. Gerstenhaber, *On dominance and varieties of commuting matrices,* Ann. Math. **73 (2)**, 324–348, (1961).

[10] F. Hirzebruch, *Some problems on differentiable and complex manifolds,* Ann. Math. **60**, 212–236 (1954).

[11] A. Iarrobino, *Hilbert scheme of points: Overview of last ten years,* In: *Algebraic Geometry: Bowdoin 1985*, Proceedings of Symposia in Pure Mathematics **46 (2)**, American Mathematical Society, Providence, RI, (1987), 297–320.

[12] F. Knop, H. Lange, *Commutative algebraic groups and intersections of quadrics*, Math. Ann. **267 (4)**, 555–571, (1984).

[13] S. Müller-Stach, *Compactifications of* $\mathbf{C}^3$ *with reducible boundary divisor*, Math. Ann. **286(1-3)**, 409–431, (1990).

[14] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory,* 3 ed., Springer-Verlag (1994).

[15] T. Oda, *Convex Bodies and Algebraic Geometry,* Springer-Verlag, (1988).

[16] T. Peternell, M. Schneider, *Compactifications of* $\mathbf{C}^n$*: a survey*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 455–466, Proc. Sympos. Pure Math., **52**, Part 2, Amer. Math. Soc., Providence, RI, (1991).

[17] D.A. Suprunenko, R.I. Tyshkevich, *Commutative Matrices*, Academic Press, New York, (1969).

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