RATIONAL POINTS ON SOME FANO CUBIC BUNDLES

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Abstract. We consider smooth Fano hypersurfaces $X_{n+2} \subset \mathbf{P}^n \times \mathbf{P}^3$ $(n \geq 1)$ given by a polynomial

$$\sum_{i=0}^{3} l_i(\mathbf{x}) y_i^3 \in \mathbf{Q}[x_0, \ldots, x_n, y_0, \ldots, y_3]$$

where $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ are homogeneous linear forms in $x_0, \ldots, x_n$. We obtain lower bounds for the number of F-rational points of bounded anticanonical height in arbitrary nonempty Zariski open subsets $U \subset X_{n+2}$ for number fields F containing $\mathbf{Q}(\sqrt{-3})$. These bounds contradict previous expectations about the distribution of F-rational points of bounded height on Fano varieties.

Points rationnels sur certains fibrés en cubiques de Fano

Résumé français. Nous considérons des hypersurfaces lisses de Fano $X_{n+2} \subset \mathbf{P}^n \times \mathbf{P}^3$ $(n \geq 1)$ données par un polynôme

$$\sum_{i=0}^{3} l_i(\mathbf{x}) y_i^3 \in \mathbf{Q}[x_0, \ldots, x_n, y_0, \ldots, y_3]$$

où $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ sont des formes linéaires homogènes en $x_0, \ldots, x_n$. Nous obtenons des bornes inférieures pour le nombre de points F-rationnels de hauteur bornée (relativement au diviseur anticanonique) dans des ouverts de Zariski non vides quelconques $U \subset X_{n+2}$ pour tous les corps de nombres F contenant $\mathbf{Q}(\sqrt{-3})$. Ces bornes vont à l'encontre de la distribution attendue des points F-rationnels de hauteur bornée sur les variétés de Fano.

Version française abrégée

Théorème. *Pour tout $n \geq 1$, l'hypersurface X_{n+2} a les propriétés suivantes*

– c'est une variété de Fano lisse contenant un ouvert de Zariski isomorphe à $\mathbf{A}^{n+2}$;

– les fibres de la projection naturelle $\pi : X_{n+2} \to \mathbf{P}^n$ au-dessus des points fermés de l'ouvert de Zariski $U_P \subset \mathbf{P}^n$ défini par l'équation $\prod_{i=0}^{3} l_i(x) \neq 0$ sont des surfaces cubiques diagonales lisses de $\mathbf{P}^3$;

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– le groupe de Picard de X_{n+2} sur tout corps contenant $\mathbf{Q}$ est isomorphe à $\mathbf{Z}\oplus\mathbf{Z}$;

– le fibré canonique sur X_{n+2} induit sur chaque fibre de π le fibré canonique de cette fibre.

Théorème. *Soit Y une surface cubique diagonale dans $\mathbf{P}^3$ de la forme*

$$b_0^3y_0^3 + \cdots + b_3^3y_3^3 = 0$$

où les b_i sont des éléments non nuls d'un corps F contenant $\mathbf{Q}(\sqrt{-3})$. Alors, il existe une constante $c > 0$ telle que, pour tout ouvert de Zariski non vide $U \subset Y$, le nombre de points F-rationnels de U dont la hauteur anticanonique est inférieure à B est minoré par $cB(\log B)^3$.

La conjecture suivante a été inspirée par la conjecture de croissance linéaire de Manin [6] et par l'extrapolation de résultats connus (méthode du cercle, variétés de drapeaux, variétés toriques) :

Conjecture. *Soit X une variété de Fano lisse sur un corps de nombres E. Alors, il existe un ouvert de Zariski $U \subset X$ et une extension finie F_0 de E, tels que pour toute extension finie F de F_0, le nombre de points F-rationnels contenus dans U dont la hauteur anticanonique est majorée par B est asymptotique à $cB(\log B)^{t-1}$ où t est le rang du groupe de Picard de X sur F et c une constante positive.*

Cette conjecture a été raffinée par E. Peyre qui a proposé une interprétation adélique de la constante c.

Les énoncés précédents prouvent que cette conjecture est fausse pour les fibrés en cubiques de Fano X_{n+2}.

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1. Cubic bundles

Let X_{n+2} be a hypersurface in $\mathbf{P}^n \times \mathbf{P}^3$ ($n \geq 1$) defined by the equation

$$P(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{3} l_i(\mathbf{x})y_i^3 = 0$$

where

$$P(\mathbf{x},\mathbf{y}) \in \mathbf{Q}[x_0, \ldots, x_n, y_0, \ldots, y_3]$$

and $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ are homogeneous linear forms in $x_0, \ldots, x_n$. Put $k = \min(n+1, 4)$. We shall always assume that any k forms among $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ are linearly independent. It is elementary to check the following statements:

Proposition 1.1. *The hypersurface X_{n+2} is a smooth Fano variety containing a Zariski open subset U_{n+2} which is isomorphic to $\mathbf{A}^{n+2}$.*

Proposition 1.2. *Let $U_P \subset \mathbf{P}^n$ be the Zariski open subset defined by the condition*

$$\prod_{i=0}^{3} l_i(\mathbf{x}) \neq 0.$$

Then the fibers of the natural projection $\pi : X_{n+2} \to \mathbf{P}^n$ over closed points of U_P are smooth diagonal cubic surfaces in $\mathbf{P}^3$.

Proposition 1.3. *The Picard group of X_{n+2} over an arbitrary field containing $\mathbf{Q}$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.*

2. Heights on cubic surfaces

Let F be a number field, $Val(F)$ the set of all valuations of F, W a projective algebraic variety over F, $W(F)$ the set of F-rational points of W, D a very ample divisor on W, and $\gamma = \{s_0, \dots, s_m\}$ a basis over F of the space of global sections $\Gamma(W, \mathcal{O}(D))$. The *height function associated with D and γ*

$$H(W, D, \gamma, x) \,:\, W(F) \to \mathbf{R}_{>0}$$

is given by the formula

$$H(W, D, \gamma, x) = \prod_{v \in Val(F)} \max_{i=0,\dots,m} |s_i(x)|_v,$$

where $|\cdot|_v : F_v \to \mathbf{R}_{>0}$ is the multiplier of a Haar measure on the additive group of the v-adic completion of F.

Definition 2.1. *Let $Z \subset W$ be a locally closed algebraic subset of W, B a positive real number. Define*

$$N(Z, D, \gamma, B) := Card\{x \in W(F) \cap Z \mid H(W, D, \gamma, x) \leq B\}.$$

The following classical statement is due to A. Weil:

Theorem 2.2. *Let $\gamma' = \{s'_0, \dots, s'_m\}$ be another basis in $\Gamma(W, \mathcal{O}(D))$. Then there exist two positive constants c_1, c_2 such that*

$$c_1 \leq \frac{H(W, D, \gamma, x)}{H(W, D, \gamma', x)} \leq c_2$$

for all $x \in W(F)$.

For a smooth projective variety W we denote by $-K_W$ the anticanonical divisor on W.

Theorem 2.3. *Let $Y \subset \mathbf{P}^3$ be a smooth cubic surface over F, $\gamma = \{s_0, s_1, s_2, s_3\}$ the basis of global sections of $\mathcal{O}(-K_Y)$ corresponding to the standard homogeneous coordinates on $\mathbf{P}^3$. Assume that Y can be obtained by blowing up of 6 F-rational points in $\mathbf{P}^2$. Then for any nonempty Zariski open subset $U \subset Y$ one has*

$$N(U, -K_Y, \gamma, B) \geq cB(\log B)^3$$

for all $B > 0$ and for some positive constant c.

Proof. Assume that Y is obtained by blowing up of $p_1, \dots, p_6 \in \mathbf{P}^2(F)$. By 2.2, we can assume without loss of generality that $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$. Denote by Y_0 the Del Pezzo surface obtained by blowing up p_1, p_2, p_3. Let $f : Y \to Y_0$ be the contraction of exceptional curves $C_4, C_5, C_6 \subset Y$ lying over p_4, p_5, p_6.

Let V be the 10-dimensional space over F of all homogeneous polynomials of degree 3 in variables z_0, z_1, z_2. We identify $\Gamma(Y, \mathcal{O}(-K_Y))$ with the subspace in V consisting of all polynomials vanishing in $p_1, \dots, p_6$. Analogously, we identify $\Gamma(Y_0, \mathcal{O}(-K_{Y_0}))$ with the subspace in V consisting of all polynomials vanishing in p_1, p_2, p_3. Let $\gamma_0 = \{s_0, \dots, s_6\} \subset V$ the extension of the basis γ to a basis of the subspace $\Gamma(Y_0, \mathcal{O}(-K_{Y_0})) \subset V$.

The surface Y_0 is an smooth equivariant compactification of the split 2-dimensional algebraic torus over F

$$(\mathbf{G}_m)^2 = \mathbf{P}^2 \setminus \{l_{12}, l_{13}, l_{23}\}$$

where l_{ij} denotes the projective line in $\mathbf{P}^2$ through p_i and p_j. Since Y_0 is a smooth toric variety, the main theorem in [3] shows that the following asymptotic formula holds:

$$N((\mathbf{G}_m)^2, -K_{Y_0}, \gamma_0', B) = c_0 B(\log B)^3(1 + o(1)), \quad B \to \infty, \tag{1}$$

where c_0 is some positive constant and

$$\gamma_0' = \{z_0z_1z_2, z_1^2z_2, z_1z_2^2, z_2^2z_0, z_2z_0^2, z_0^2z_1, z_0z_1^2\}.$$

Let U be any nonempty Zariski open subset in Y. We denote by U_0 a nonempty open subset in U such that the restriction of f on U_0 is an isomorphism and $f(U_0)$ is contained in $(\mathbf{G}_m)^2 \subset Y_0$. Since

$$\prod_{v \in Val(F)} \max_{i=0,\dots,3} |s_i(x)|_v \leq \prod_{v \in Val(F)} \max_{i=0,\dots,6} |s_i(x)|_v$$

holds for every F-rational point $x \in U_0$, we obtain

$$N(U_0, -K_Y, \gamma, B) \geq N(U_0, -K_{Y_0}, \gamma_0, B) \tag{2}$$

for any $B > 0$. By 2.2, there exists a positive constant c_3 such that

$$N(U_0, -K_{Y_0}, \gamma_0, B) \geq c_3 N(U_0, -K_{Y_0}, \gamma_0', B). \tag{3}$$

On the other hand,

$$N(U_0, -K_{Y_0}, \gamma'_0, B) = N((\mathbf{G}_m)^2, -K_{Y_0}, \gamma'_0, B) - N(Z, -K_{Y_0}, \gamma'_0, B), \tag{4}$$

where $Z = (\mathbf{G}_m)^2 \setminus U_0$. Let $Z_1, \dots, Z_l$ be the irreducible components of Z, $\overline{Z}_i$ the closure of Z_i in Y_0 ($i = 1, \dots, l$). It is known that

$$N(Z_i, -K_{Y_0}, \gamma'_0, B) \leq c_4 B^{2/(deg\,\overline{Z}_i)} \tag{5}$$

holds for some positive constant c_4, where $deg\,\overline{Z}_i$ denotes the degree of $\overline{Z}_i$ with respect to the anticanonical divisor $-K_{Y_0}$. Since every irreducible curve $C \subset Y_0$ with $deg\,C = 1$ is a component of $Y_0 \setminus (\mathbf{G}_m)^2$, we have $deg\,\overline{Z}_i \geq 2$; i.e.,

$$N(Z_i, -K_{Y_0}, \gamma'_0, B) \leq c_4 B \tag{6}$$

holds for all $i = 1, \dots, l$.

It follows from the asymptotic formula (1) combined with (2), (3) and (4) that there exists a positive constant c such that

$$N(U_0, -K_Y, \gamma, B) \geq cB(\log B)^3$$

holds for all $B > 0$. This yields the statement, since U_0 is contained in U. □

Corollary 2.4. *Let Y be a smooth diagonal cubic surface in $\mathbf{P}^3$ defined by the equation*

$$a_0 y_0^3 + a_1 y_1^3 + a_2 y_2^3 + a_3 y_3^3 = 0$$

with coefficients $a_0, \dots, a_3$ in a number field F which contains $\mathbf{Q}(\sqrt{-3})$. Assume that there exist numbers $b_0, \dots, b_3 \in F^*$ such that $a_i = b_i^3$ ($i = 0, \dots, 3$). Then for any nonempty Zariski open subset $U \subset Y$ one has

$$N(U, -K_Y, \gamma, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proof. It follows from our assumptions on the coefficients $a_0, \dots, a_3$ and on the field F that all 27 lines on Y are defined over F. Hence, Y can be obtained from $\mathbf{P}^2$ by blowing up of 6 F-rational points. Now the statement follows from 2.3. □

3. Rational points on X_{n+2}

We have the natural isomorphism

$$\Gamma(X_{n+2}, \mathcal{O}(-K_{X_{n+2}})) \cong \Gamma(\mathbf{P}^3, \mathcal{O}(1)) \otimes \Gamma(\mathbf{P}^n, \mathcal{O}(n)).$$

We remark that the canonical line bundle on X_{n+2} induces the canonical line bundle in the fibers of π. Let $\{t_0, \dots, t_m\}$ be a basis in $\Gamma(\mathbf{P}^n, \mathcal{O}(n))$ and $\{s_0, s_1, s_2, s_3\}$ the standard basis in $\Gamma(\mathbf{P}^3, \mathcal{O}(1))$. Denote by γ the basis of $\Gamma(X_{n+2}, \mathcal{O}(-K_{X_{n+2}}))$ consisting of $s_i \otimes t_j$ ($i = 0, \dots, 3;\ j = 0, \dots, m$).

Theorem 3.1. *Let $n \geq 3$. Then for any nonempty Zariski open subset $U \subset X_{n+2}$ and for any field F containing $\mathbf{Q}(\sqrt{-3})$, one has*

$$N(U, -K_{X_{n+2}}, \gamma, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proof. Consider the projection $\pi : U \to \mathbf{P}^n$. Let U' be a Zariski open subset in $\pi(U) \cap U_P$ such that the image of U' under the dominant (rational) mapping

$$\psi \ : \ \mathbf{P}^n - - \to \mathbf{P}^3$$

$$\psi(x_0 : \ldots : x_n) = (l_0(\mathbf{x}) : \ldots : l_3(\mathbf{x}))$$

is Zariski open in $\mathbf{P}^3$. Denote by φ the finite morphism

$$\varphi \ : \ \mathbf{P}^3 \to \mathbf{P}^3,$$

$$\varphi(z_0 : \ldots : z_3) = (z_0^3 : \ldots : z_3^3).$$

Since $\mathbf{P}^3(F)$ is Zariski dense in $\mathbf{P}^3$, there is a point $p \in \mathbf{P}^3(F) \cap \varphi^{-1}(\psi(U'))$. Since $U'(F) \cap \psi^{-1}(\varphi(p))$ is Zariski dense in $\psi^{-1}(\varphi(p))$, there exists $q \in U'(F) \cap \psi^{-1}(\varphi(p))$. Therefore, the fiber of π over q is a diagonal cubic surface Y_q and $U \cap Y_q \subset Y_q$ is a nonempty Zariski open subset. It remains to apply 2.4. □

Theorem 3.2. *Let $n = 2$. Then there exists a number field F_0 depending only on X_{n+2} such that for any nonempty Zariski open subset $U \subset X_{n+2}$ for any field F containing F_0 one has*

$$N(U, -K_{X_{n+2}}, \gamma, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proof. Let U' be a Zariski open subset in $\pi(U) \cap U_P$. We have the linear embedding

$$\psi \ : \ \mathbf{P}^2 \hookrightarrow \mathbf{P}^3$$

defined by $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$. Then $\varphi^{-1}(\psi(\mathbf{P}^2))$ is a smooth diagonal cubic surface $S \subset \mathbf{P}^3$ defined over $\mathbf{Q}$. Let F_0 be a finite extension of $\mathbf{Q}(\sqrt{-3})$ such that $S(F_0)$ is Zariski dense in S. Then there exists a point $p \in S(F_0)$ such that $q = \varphi(p)$ is contained in U'. Therefore, the fiber of π over q is a diagonal cubic surface Y_q, and $U \cap Y_q \subset Y_q$ is a nonempty Zariski open subset. It remains to apply 2.4. □

Theorem 3.3. *Let $n = 1$. Then for any nonempty Zariski open subset $U \subset X_{n+2}$, there exists a number field F_0 (which depends on U) such that for any field F containing F_0, one has*

$$N(U, -K_{X_{n+2}}, \gamma, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proof. Let U' be a Zariski open subset in $\pi(U) \cap U_P$. We have the linear embedding

$$\psi \;:\; \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$$

defined by $l_0(\mathbf{x}), \dots , l_3(\mathbf{x})$. Then $\varphi^{-1}(\psi(\mathbf{P}^1))$ is an algebraic curve $C \subset \mathbf{P}^3$ which is a complete intesection of two diagonal cubic surfaces defined over $\mathbf{Q}$. Let F_0 be a finite extension of $\mathbf{Q}(\sqrt{-3})$ such that there exists an F_0-rational point $p \in C(F_0) \cap \varphi^{-1}(U')$. Then the fiber of π over $q = \varphi(p)$ is a diagonal cubic surface Y_q and $U \cap Y_q \subset Y_q$ is a nonempty Zariski open subset. It remains to apply 2.4. □

4. Conclusions

The following statement, inspired by the Linear Growth conjecture of Manin ([6]) and by extrapolation of known results (circle method, flag varieties, toric varieties), has been expected to be true [1, 4]:

Conjecture 4.1. *Let X be a smooth Fano variety over a number field E. Then there exist a Zariski open subset $U \subset X$ and a finite extension F_0 of E such that for all number fields F containing F_0 the following asymptotic formula holds*

$$N(U, -K_X, \gamma, B) = cB(\log B)^{t-1}(1 + o(1)), \quad B \to \infty,$$

where t is the rank of the Picard group of X over F.

Some lower and upper bounds for $N(U, -K_X, \gamma, B)$ for Del Pezzo surfaces and Fano threefolds have been obtained in [6, 7].

The conjecture 4.1 was refined by E. Peyre who proposed an adelic interpretation for the constant c introducing Tamagawa numbers of Fano varieties [8]. This refined version of the conjecture has been proved for toric varieties in [2, 3].

The statements in Theorems 3.1, 3.2, 3.3 and the property 1.3 show that Conjecture 4.1 is not true for Fano cubic bundles X_{n+2} $(n \geq 1)$.

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