Manin's conjecture for toric varieties

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Abstract

We prove an asymptotic formula conjectured by Manin for the number of K-rational points of bounded height with respect to the anticanonical line bundle for arbitrary smooth projective toric varieties over a number field K.

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Contents

Introduction

Let X be a projective algebraic variety defined over a number field K, $X(K)$ the set of K-rational points of X. We are interested in relations between the geometry of X and diophantine properties of $X(K)$ in the situation when $X(K)$ is infinite. The main object of our study is the *height function* on $X(K)$ with respect to a metrized line bundle $\mathcal{L}$. A metrized line bundle is a pair $\mathcal{L} = (L, \|\cdot\|_v)$, consisting of a line bundle L equipped with a family $\{\|\cdot\|_v\}$ of v-adic metrics (v runs over the set $\mathrm{Val}(K)$ of all valuation of K), satisfying certain conditions. This defines a height function $H_{\mathcal{L}} : X(K) \to \mathbf{R}_{>0}$ by

$$H_{\mathcal{L}}(x) := \prod_{v \in \mathrm{Val}(K)} \|f(x)\|_v^{-1}$$

where f is a K-rational local section of the line bundle L not vanishing in $x \in X(K)$.

Let $\mathrm{Pic}(X)$ be the Picard group of X and $\Lambda_{\mathrm{eff}} \subset \mathrm{Pic}(X)$ the cone of effective divisors. Assume that the class of L is contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}} \subset \mathrm{Pic}(X)_{\mathbf{R}}$. In this case, some positive tensor power of L defines a birational map of X into some projective space.

We denote by $U_L \subset X$ the Zariski open subset such that the restriction of the above birational map to U_L is an isomorphism on its image.

For any Zariski open subset $U \subset U_L$ consider the *height zeta-function* defined by the following series [1, 9]:

$$Z_{\mathcal{L},U}(s) = \sum_{x \in U(K)} H_{\mathcal{L}}(x)^{-s}.$$

Then $Z_{\mathcal{L},U}(s)$ converges absolutely and uniformly for $\mathrm{Re}(s) \gg 0$. A Tauberian theorem relates the analytic properties of the zeta-function with the asymptotic behaviour of the number $N(U, \mathcal{L}, B)$ of K-rational points $x \in U(K)$ with $H_{\mathcal{L}}(x) \leq B$ as $B \to \infty$.

We define

$$\alpha(\mathcal{L}, U) = \inf\{a \in \mathbf{R} \mid Z_{\mathcal{L},U}(s) \text{ converges for } \mathrm{Re}(s) > a\}.$$

We say that U does not contain *$\mathcal{L}$-accumulating subvarieties* if for any nonempty Zariski open subset $U' \subset U$ we have $\alpha(\mathcal{L}, U) = \alpha(\mathcal{L}, U')$.

One expects a good accordance between the geometry of X and diophantine properties of the set of K-rational points of X which are contained in the complement $U \subset X$ to some proper closed subvarieties in X. Denote by $\mathcal{K}^{-1}$ the metrized anicanonical line bundle. The following conjecture is due to Yu. I. Manin [9]:

Let X be a smooth projective variety over a number field K whose anticanonical line bundle is ample (i.e., X is a Fano variety). Assume that the set $X(K)$ of K-rational points is Zariski dense. Let $U \subset X$ be the largest Zariski open subset which doesn't contain $\mathcal{K}^{-1}$-accumulating subvarieties. Then

$$N(U, \mathcal{K}^{-1}, B) = c(X, \mathcal{K}^{-1}, K) B (\log B)^{k-1}(1 + o(1)) \text{ for } B \to \infty$$

where k equals the rank of the Picard group $\mathrm{Pic}(X)$ over K and $c(X, \mathcal{K}^{-1}, K)$ is some positive constant which depends on X, K and the choice of v-adic metrics on the anticanonical line bundle.

The above conjecture was refined by E. Peyre [21] who defined Tamagawa numbers of Fano varieties and proposed an interpretation of $c(X, \mathcal{K}^{-1}, K)$ in terms of these numbers.

Unfortunately, the conjecture of Manin is not true in general (see [3]). It has been proved for some class of Fano varieties, e.g., for generalized flag varieties, complete intersections of small degree and for some blow ups of projective spaces [9, 1, 21].

In this paper we prove Manin's conjecture and compute the constant $c(X, \mathcal{K}^{-1}, K)$ for arbitrary smooth projective equivariant compactifications of algebraic tori over number fields, i.e., for toric varieties $\mathbf{P}_\Sigma$ associated with a Galois invariant finite polyhedral fan Σ [27]. We restrict ourselves to the case when U is the dense torus orbit. It is easy to show that U doesn't contain $\mathcal{L}$-accumulating subvarieties for any metrized line bundle $\mathcal{L}$. On the other hand, it might happen that there is some larger Zariski open subset $U' \subset \mathbf{P}_\Sigma$ which contains U and which does not contain $\mathcal{L}$-accumulating subvarieties. It is possible to show that the the asymptotic formula for $N(U', \mathcal{K}^{-1}, B)$ does not depend on the choice of such a Zariski open U', but we decided to postpone the proof of this fact.

One of our main ideas for the computation of the height zeta-function on a toric variety $\mathbf{P}_\Sigma$ is to introduce some canonical simultaneous metrizations on all line bundles and to obtain a pairing

$$H_\Sigma(x, \mathbf{s}) : \; T(K) \times \mathrm{Pic}(\mathbf{P}_\Sigma)_{\mathbf{C}} \to \mathbf{C}$$

between the set of rational points $T(K) \subset \mathbf{P}_\Sigma(K)$ in the Zariski open subset T and the complexified Picard group, extending the usual height pairing between $T(K)$ and $\mathrm{Pic}(\mathbf{P}_\Sigma)$. This allows to extend the one-parameter zeta-function to a function $Z_\Sigma(\mathbf{s})$ defined on the complexified Picard group $\mathrm{Pic}(\mathbf{P}_\Sigma)_{\mathbf{C}}$ and holomorphic when the $\mathrm{Re}(\mathbf{s})$ is contained in the interior of the cone $[\mathcal{K}^{-1}] + \Lambda_{\mathrm{eff}}$, where $\Lambda_{\mathrm{eff}} \subset \mathrm{Pic}(\mathbf{P}_\Sigma)_{\mathbf{R}}$ is the cone of effective divisors of $\mathbf{P}_\Sigma$.

The second step is to use the multiplicative group structure on the torus $T \subset \mathbf{P}_\Sigma$. With our choice of metrics, the height zeta-function becomes a function on the adelic group $T(\mathbf{A}_K)$ invariant under the closed subgroup $T(K)\mathbf{K}_T$, where $\mathbf{K}_T \subset T(\mathbf{A}_K)$ is the maximal compact subgroup. The key idea is to use the *Poisson formula* in order to obtain an integral representation for $Z_\Sigma(\mathbf{s})$.

Our third step is the study of analytic properties of $Z_\Sigma(\mathbf{s})$ using the above integral formula and properties of $\mathcal{X}$-functions of convex cones.

In section 1 we introduce notations and basic notions from the theory of toric varieties over non-closed fields.

In section 2 we recall the definitions of Tamagawa numbers of algebraic tori and Tamagawa numbers of algebraic varieties with metrized anticanonical bundle.

In section 3 we define simultaneous metrizations of all line bundles on toric varieties, introduce the height zeta-function and give formulas for local Fourier transforms of heights.

In section 4 we prove the Poisson formula which yields an integral representation of the height zeta-function.

In section 5 we formulate basic properties of $\mathcal{X}$-functions of convex finitely generated polyhedral cones.

In section 6 we prepare the necessary analytic tools.

And finally, in section 7 we prove our main theorem:

Let $\mathbf{P}_\Sigma$ be a smooth projective compactification of an algebraic torus T over K. Let k be the rank of $\mathrm{Pic}(\mathbf{P}_\Sigma)$. Then there is only a finite number $N(T, \mathcal{K}^{-1}, B)$ of K-rational points $x \in T(K)$ having the anticanonical height $H_{\mathcal{K}^{-1}}(x) \leq B$. Moreover,

$$N(T, \mathcal{K}^{-1}, B) = \frac{\Theta(\Sigma)}{(k-1)!} \cdot B(\log B)^{k-1}(1 + o(1)), \quad B \to \infty,$$

with the constant $\Theta(\Sigma) = \alpha(\mathbf{P}_\Sigma)\beta(\mathbf{P}_\Sigma)\tau_{\mathcal{K}}(\mathbf{P}_\Sigma)$, where:

1. *$\alpha(\mathbf{P}_\Sigma)$ is a constant defined by the geometry of the cone of effective divisors $\Lambda_{\text{eff}} \subset \mathrm{Pic}(\mathbf{P}_\Sigma)_{\mathbf{R}}$;*
2. *$\beta(\mathbf{P}_\Sigma)$ is the order of the non-trivial part of the Brauer group of $\mathbf{P}_\Sigma$;*
3. *$\tau_{\mathcal{K}}(\mathbf{P}_\Sigma)$ is the Tamagawa number associated with the metrized canonical sheaf on $\mathbf{P}_\Sigma$ (as it was defined by E.Peyre in [21]).*

Our results provide first examples for asymptotics on unirational varieties which are not rational and on varieties without weak approximation, in general. A new phenomenon is the appearance of the non-trivial part of the Brauer group $Br(\mathbf{P}_\Sigma)/Br(K)$ in the asymptotic constant. We don't need to assume that the anticanonical class of $\mathbf{P}_\Sigma$ is ample (i.e., $\mathbf{P}_\Sigma$ is a toric Fano variety).

This paper is a continuation of our paper [2], where we proved the conjecture of Manin about the distribution of K-rational points of bounded height for the case of projective compactifications of anisotropic tori. An equvariant compactification $\mathbf{P}_\Sigma$ of an anisotropic torus T is much simpler in many aspects: the cone of effective divisors Λ_{eff} is always simplicial, all K-rational points of $\mathbf{P}_\Sigma$ are contained in T, and the group $T(\mathbf{A}_K)/T(K)$ is compact (this last property significantly simplifies the Poisson formula).

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1 Algebraic tori and toric varieties

Let X_K be an algebraic variety defined over a number field K and E/K a finite extension of number fields. We will denote the set of E-rational points of X_K by $X(E)$ and by X_E the E-variety obtained from X_K by base change. We sometimes omit the subscript in X_E if the respective field of definition is clear from the context. Let $\mathbf{G}_{m,E} = \mathrm{Spec}(E[x, x^{-1}])$ be the multiplicative group scheme over E.

Definition 1.1 A linear algebraic group T_K is called a *d-dimensional algebraic torus* if there exists a finite extension E/K such that T_E is isomorphic to $(\mathbf{G}_{m,E})^d$. The field E is called the *splitting field* of T.

For any field E we denote by $\hat{T}_E = \mathrm{Hom}\,(T, E^*)$ the group of regular E-rational characters of T.

Theorem 1.2 [8, 19, 26] *There is a contravariant equivalence between the category of algebraic tori defined over a number field K and the category of torsion free* $\mathrm{Gal}(E/K)$*-modules of finite rank over* $\mathbf{Z}$*. The functors are given by*

$$M \to T = \mathrm{Spec}(K[M]);\;\; T \to \hat{T}_E.$$

The above contravariant equivalence is functorial under field extensions of K.

Let $\mathrm{Val}(K)$ be the set of all valuations of a global field K. Denote by S_∞ the set of archimedian valuations of K. For any $v \in \mathrm{Val}(K)$, we denote by K_v the completion of K with respect to v. Let E be a finite Galois extension of K with the Galois group G. Let $\mathcal{V}$ be an extension of v to E, $E_{\mathcal{V}}$ the completion of E with respect to $\mathcal{V}$. Then

$$\mathrm{Gal}(E_{\mathcal{V}}/K_v) \cong G_v \subset G,$$

where G_v is the decomposition subgroup of G and $K_v \otimes_K E \cong \prod_{\mathcal{V}|v} E_{\mathcal{V}}$. Let T be an algebraic torus over K with the splitting field E. Denote by $T(K_v) =$ the v-adic completion of $T(K)$ and by $T(\mathcal{O}_v) \subset T(K_v)$ its maximal compact subgroup.

Definition 1.3 Denote by $T(\mathbf{A}_K)$ the adele group of T. Define

$$T^1(\mathbf{A}_K) = \{\mathbf{t} \in T(\mathbf{A}_K) : \prod_{v \in \mathrm{Val}(K)} \mid m(t_v) \mid_v = 1, \text{ for all } m \in \hat{T}_K \subset M\}.$$

Let

$$\mathbf{K}_T = \prod_{v \in \mathrm{Val}(K)} T(\mathcal{O}_v),$$

be the maximal compact subgroup of $T(\mathbf{A}_K)$.

Proposition 1.4 [19] *The groups $T(\mathbf{A}_K)$, $T^1(\mathbf{A}_K)$, $T(K)$, $\mathbf{K}_T$ have the following properties:*

(i) $T(\mathbf{A}_K)/T^1(\mathbf{A}_K) \cong \mathbf{R}^t$, *where t is the rank of $\hat{T}_K$;*

(ii) $T^1(\mathbf{A}_K)/T(K)$ *is compact;*

(iii) $T^1(\mathbf{A}_K)/T(K) \cdot \mathbf{K}_T$ *is isomorphic to the direct product of a finite group $\mathbf{cl}(T_K)$ and a connected compact abelian topological group which dimension equals the rank r' of the group of $\mathcal{O}_K$-units in $T(K)$;*

(iv) $W(T) = \mathbf{K}_T \cap T(K)$ *is a finite group of all torsion elements in $T(K)$.*

Definition 1.5 We define the following cohomological invariants of the algebraic torus T:

$$h(T) = \mathrm{Card}[H^1(G, M)],$$

$$\mathrm{III}(T) = \mathrm{Ker}\,[H^1(G, T(E)) \to H^1(G_v, T(\mathbf{A}_E))],$$

$$i(T) = \mathrm{Card}[\mathrm{III}(T)].$$

Definition 1.6 Let $\overline{T(K)}$ be the closure of $T(K)$ in $T(\mathbf{A}_K)$ in the *direct product topology*. Define the *obstruction group to weak approximation* as

$$A(T) = T(\mathbf{A}_K)/\overline{T(K)}.$$

Remark 1.7 It is known that over the splitting field E one has $A(T_E) = 0$.

Let T_K be a d-dimensional algebraic torus over K with splitting field E and $G = \mathrm{Gal}(E/K)$. Denote by M the lattice $\hat{T}_E$ and by $N = \mathrm{Hom}(M, \mathbf{Z})$ the dual abelian group. Let us recall standard facts about toric varieties over arbitrary fields [5, 6, 10, 18, 2].

Definition 1.8 A finite set Σ consisting of convex rational polyhedral cones in $N_{\mathbf{R}} = N \otimes \mathbf{R}$ is called a *d-dimensional fan* if the following conditions are satisfied:

(i) every cone $\sigma \in \Sigma$ contains $0 \in N_{\mathbf{R}}$;
(ii) every face σ' of a cone $\sigma \in \Sigma$ belongs to Σ;
(iii) the intersection of any two cones in Σ is a face of both cones.

Definition 1.9 A d-dimensional fan Σ is called *complete and regular* if the following additional conditions are satisfied:

(i) $N_{\mathbf{R}}$ is the union of cones from Σ;
(ii) every cone $\sigma \in \Sigma$ is generated by a part of a $\mathbf{Z}$-basis of N.

We denote by $\Sigma(j)$ the set of all j-dimensional cones in Σ. For each cone $\sigma \in \Sigma$ we denote by $N_{\sigma,\mathbf{R}}$ the minimal linear subspace containing σ.

Theorem 1.10 *A complete regular d-dimensional fan Σ defines a smooth equivariant compactification $\mathbf{P}_{\Sigma,E}$ of the E-split algebraic torus T_E. The* toric variety $\mathbf{P}_{\Sigma,E}$ *has the following properties:*

(i) There is a T_E-invariant open covering by affine subsets $U_{\sigma,E}$:

$$\mathbf{P}_{\Sigma,E} = \bigcup_{\sigma \in \Sigma} U_{\sigma,E}.$$

The affine subsets are defined as $U_{\sigma,E} = \mathrm{Spec}(E[M \cap \check{\sigma}])$, where $\check{\sigma}$ is the cone in $M_{\mathbf{R}}$ which is dual to σ.

(ii) There is a representation of $\mathbf{P}_{\Sigma,E}$ *as a disjoint union of split algebraic tori* $T_{\sigma,E}$ *of dimension* $\dim T_{\sigma,E} = d - \dim \sigma$*, that is, for all fields* F/E *we have*

$$\mathbf{P}_{\Sigma,E}(F) = \bigcup_{\sigma \in \Sigma} T_{\sigma,E}(F).$$

For each j*-dimensional cone* $\sigma \in \Sigma(j)$ *we denote by* $T_{\sigma,E}$ *the kernel of a homomorphism* $T_E \to (\mathbf{G}_{m,E})^j$ *defined by a* $\mathbf{Z}$*-basis of the sublattice* $N \cap N_{\sigma,\mathbf{R}} \subset N$*.*

To construct compactifications of non-split tori T_K over K, we need a complete fan Σ of cones having an additional combinatorial structure: an *action of the Galois group* $G = \mathrm{Gal}(E/K)$ [27]. The lattice $M = \hat{T}_E$ is a G-module and we have a representation $\rho : G \to \mathrm{Aut}(M)$. Denote by ρ^* the induced dual representation of G in $\mathrm{Aut}(N) \cong \mathrm{GL}(d, \mathbf{Z})$.

Definition 1.11 A complete fan $\Sigma \subset N_{\mathbf{R}}$ is called *G-invariant* if for any $g \in G$ and for any $\sigma \in \Sigma$, one has $\rho^*(g)(\sigma) \in \Sigma$. Let N^G (resp. M^G, $N^G_{\mathbf{R}}$, $M^G_{\mathbf{R}}$ and Σ^G) be the subset of G-invariant elements in N (resp. in M, $N \otimes \mathbf{R}$, $M \otimes \mathbf{R}$ and Σ). Denote by $\Sigma_G \subset N^G_{\mathbf{R}}$ the fan consisting of all possible intersections $\sigma \cap N^G_{\mathbf{R}}$ where σ runs over all cones in Σ.

The following theorem is due to Voskresenskiî [27]:

Theorem 1.12 *Let* Σ *be a complete regular* G*-invariant fan in* $N_{\mathbf{R}}$*. Assume that the complete toric variety* $\mathbf{P}_{\Sigma,E}$ *defined over the splitting field* E *by the* G*-invariant fan* Σ *is projective. Then there exists a unique complete algebraic variety* $\mathbf{P}_{\Sigma,K}$ *over* K *such that its base extension* $\mathbf{P}_{\Sigma,K} \otimes_{\mathrm{Spec}(K)} \mathrm{Spec}(E)$ *is isomorphic to the toric variety* $\mathbf{P}_{\Sigma,E}$*. The above isomorphism respects the natural* G*-actions on* $\mathbf{P}_{\Sigma,K} \otimes_{\mathrm{Spec}(K)} \mathrm{Spec}(E)$ *and* $\mathbf{P}_{\Sigma,E}$*.*

Remark 1.13 Our definition of heights and the proof of the analytic properties of height zeta functions do not use the projectivity of respective toric varieties. We note that there exist non-projective compactifications of split algebraic tori. We omit the technical discussion of non-projective compactifications of non-split tori.

We proceed to describe the algebraic geometric structure of the variety $\mathbf{P}_{\Sigma,K}$ in terms of the fan with Galois-action. Let $\mathrm{Pic}(\mathbf{P}_{\Sigma,K})$ be the Picard group and Λ_{eff} the cone in $\mathrm{Pic}(\mathbf{P}_{\Sigma,K})$ generated by classes of effective divisors. Let $\mathcal{K}$ be the canonical line bundle of $\mathbf{P}_{\Sigma,K}$.

Definition 1.14 A continuous function $\varphi : N_{\mathbf{R}} \to \mathbf{R}$ is called Σ-*piecewise linear* if its restriction to every cone $\sigma \in \Sigma$ is a linear function; i.e., for every cone $\sigma \in \Sigma$ there exists an element $m_\sigma \in M_{\mathbf{R}}$ such that $\varphi|_\sigma(x) =< x, m_\sigma >$ where $< *, * >: N_{\mathbf{R}} \times M_{\mathbf{R}} \to \mathbf{R}$ is the pairing induced from the duality between N and M. It is called *integral* if $\varphi(N) \subset \mathbf{Z}$. Denote the group of Σ-piecewise linear integral functions by $PL(\Sigma)$.

We see that the G-action on M (and N) induces a G-action on the free abelian group $PL(\Sigma)$. Denote by $e_1, \ldots, e_n$ the primitive integral generators of all 1-dimensional cones in Σ. A function $\varphi \in PL(\Sigma)$ is determined by its values on e_i, $(i = 1, ..., n)$. Let T_i be the $(d-1)$-dimensional torus orbit corresponding to the cone $\mathbf{R}_{\geq 0} e_i \in \Sigma(1)$ and $\overline{T}_i$ the Zariski closure of T_i in $\mathbf{P}_{\Sigma,E}$.

Proposition 1.15 *Let $\mathbf{P}_{\Sigma,K}$ be a smooth toric variety over K which is an equivariant compactification of an algebraic torus T_K with splitting field E and Σ the corresponding complete regular fan with $G = \mathrm{Gal}(E/K)$-action. Then:*

(i) There is an exact sequence

$$0 \to M^G \to PL(\Sigma)^G \to \mathrm{Pic}(\mathbf{P}_{\Sigma,K}) \to H^1(G, M) \to 0.$$

(ii) Let

$$\Sigma(1) = \Sigma_1(1) \cup ... \cup \Sigma_r(1)$$

be the decomposition of $\Sigma(1)$ into a union of G-orbits. The cone of effective divisors Λ_{eff} is generated by classes of G-invariant divisors

$$D_j = \sum_{\mathbf{R}_{\geq 0} e_i \in \Sigma_j(1)} \overline{T}_i \ \ (j = 1, ..., r).$$

(iii) The class of the anticanonical line bundle $\mathcal{K}^{-1} \in \mathrm{Pic}(\mathbf{P}_{\Sigma,K})$ is the class of the G-invariant piecewise linear function $\varphi_\Sigma \in PL(\Sigma)^G$ given by $\varphi_\Sigma(e_j) = 1$ for all $j = 1, ..., n$.

Proof. (i) It is known that $\mathrm{Pic}(\mathbf{P}_{\Sigma,K}) = (\mathrm{Pic}(\mathbf{P}_{\Sigma,E}))^G$ ([26] Prop. 4.40). It remains to take G-invariants in the standard short exact sequence

$$0 \to M \to PL(\Sigma) \to \mathrm{Pic}(\mathbf{P}_{\Sigma,E}) \to 0$$

describing the Picard group of a toric variety over a splitting field (see [10] 3.4) and notice that $H^1(G, PL(\Sigma)) = 0$, because $PL(\Sigma)$ is a permutational G-module.

(ii) The statement was proved in [2] 1.3.

(iii) The statement is standard [5, 18]

Theorem 1.16 [26, 4] *Let T be an algebraic torus over K with splitting field E. Let $\mathbf{P}_{\Sigma,K}$ be a complete smooth equivariant compactification of T. There is an exact sequence:*

$$0 \to A(T) \to Hom(H^1(G, \mathrm{Pic}(\mathbf{P}_{\Sigma,E})), \mathbf{Q}/\mathbf{Z}) \to \text{Ш}(T) \to 0.$$

Remark 1.17 The group $H^1(G, \mathrm{Pic}(\mathbf{P}_{\Sigma,E}))$ is canonically isomorphic to the non-trivial part of the Brauer group $\mathrm{Br}(\mathbf{P}_{\Sigma,K})/\mathrm{Br}(K)$, where $\mathrm{Br}(\mathbf{P}_{\Sigma,K}) = H^2_{\mathrm{et}}(\mathbf{P}_{\Sigma,K}, \mathbf{G}_m)$. This group appears as the obstruction group to the Hasse principle and weak approximation in [14, 4].

Corollary 1.18 Let $\beta(\mathbf{P}_\Sigma)$ be the cardinality of $H^1(G, \mathrm{Pic}(\mathbf{P}_{\Sigma,E}))$. Then

$$\mathrm{Card}[A(T)] = \frac{\beta(\mathbf{P}_\Sigma)}{i(T)}.$$

2 Tamagawa numbers

In this section we recall the definitions of Tamagawa numbers of tori following A. Weil [28] and of algebraic varieties with a metrized canonical line bundle following E. Peyre [21]. The constructions of Tamagawa numbers depend on a choice of a finite set of valuations $S \subset \mathrm{Val}(K)$ containing archimedian valuations and places of bad reduction, but the Tamagawa numbers themselves do not depend on S.

Let X be a smooth algebraic variety over K, $X(K_v)$ the set of K_v-rational points of X. Then a choice of local analytic coordinates $x_1, \ldots, x_d$ on $X(K_v)$ defines a homeomorphism $\phi : U \to K_v^d$ in v-adic topology between an open subset $U \subset X(K_v)$ and $\phi(U) \subset K_v^d$. Let $dx_1 \cdots dx_d$ be the Haar measure on K_v^d normalized by the condition

$$\int_{\mathcal{O}_v^d} dx_1 \cdots dx_d = \frac{1}{(\sqrt{\delta_v})^d}$$

where δ_v is the absolute different of K_v. Denote by $dx_1 \wedge \cdots \wedge dx_d$ the standard differential form on K_v^d. Then $f = \phi^*(dx_1 \wedge \cdots \wedge dx_d)$ is a local analytic section of the canonical sheaf $\mathcal{K}$. If $\| \cdot \|$ is a v-adic metric on $\mathcal{K}$, then we obtain the v-adic measure on U by the formula

$$\int_{U'} \omega_{\mathcal{K},v} = \int_{\phi(U')} \|f(\phi^{-1}(x))\|_v^{-1} dx_1 \cdots dx_d,$$

where U' is arbitrary open subset in U. The measure $\omega_{\mathcal{K},v}$ does not depend on the choice of local coordinates and extends to a global measure on $X(K_v)$ [21].

Definition 2.1 [19] Let T be an algebraic torus defined over a number field K with splitting field E. Denote by

$$L_S(s, T; E/K) = \prod_{v \in \mathrm{Val}(K)} L_v(s, T; E/K)$$

the Artin L-function corresponding to the representation

$$\rho \;:\; G = \mathrm{Gal}(E/K) \to \mathrm{Aut}(\hat{T}_E)$$

and a finite set $S \subset \mathrm{Val}(K)$ containing all archimedian valuations and all non-archimedian valuations of K which are ramified in E. By definition, $L_v(s, T; E/K) \equiv 1$ if $v \in S$, $L_v(s, T; E/K) = \det(Id - q_v^{-s} F_v)^{-1}$ if $v \notin S$, where $F_v \in \mathrm{Aut}(\hat{T}_E)$ is a representative of the Frobenius automorphism.

Let T be an algebraic torus of dimension d and Ω a T-invariant algebraic K-rational differential d-form. The form Ω defines an isomorphism of the canonical sheaf on T with the structure sheaf on T. Since the structure sheaf has a canonical metrization, using the above construction, we obtain a v-adic measure $\omega_{\Omega,v}$ on $T(K_v)$. Moreover, according to A. Weil [28], we have

$$\int_{T(\mathcal{O}_v)} \omega_{\Omega,v} = \frac{\mathrm{Card}[T(k_v)]}{q_v^d} = L_v(1, T; E/K)^{-1}$$

for all $v \notin S$. We put $d\mu_v = L_v(1,T;E/K)\omega_{\Omega,v}$ for all $v \in \mathrm{Val}(K)$. Then the local measures $d\mu_v$ satisfy

$$\int_{T(\mathcal{O}_v)} d\mu_v = 1$$

for all $v \notin S$.

Definition 2.2 We define the *canonical measure* on the adele group $T(\mathbf{A}_K)$

$$\omega_{\Omega,S} = \prod_{v\in\mathrm{Val}(K)} L_v(1,T;E/K)\omega_{\Omega,v} = \prod_{v\in\mathrm{Val}(K)} d\mu_v.$$

By the product formula, $\omega_{\Omega,S}$ does not depend on the choice of Ω. Let $\mathbf{dx}$ be the standard Lebesgue measure on $T(\mathbf{A}_K)/T^1(\mathbf{A}_K)$. There exists a unique Haar measure $\omega^1_{\Omega,S}$ on $T^1(\mathbf{A}_K)$ such that $\omega^1_{\Omega,S}\mathbf{dx} = \omega_{\Omega,S}$.

We proceed to define *Tamagawa measures* on algebraic varieties following E. Peyre [21]. Let X be a smooth projective algebraic variety over K with a metrized canonical sheaf $\mathcal{K}$. We assume that X satisfies the conditions $h^1(X,\mathcal{O}_X) = h^2(X,\mathcal{O}_X) = 0$. Under these assumptions, the Néron-Severi group $NS(X)$ (or, equivalently, the Picard group $\mathrm{Pic}(X)$ modulo torsion) over the algebraic closure $\overline{K}$ is a discrete continuous $\mathrm{Gal}(\overline{K}/K)$-module of finite rank over $\mathbf{Z}$. Denote by T_{NS} the corresponding torus under the duality from 1.2 and by E_{NS} a splitting field.

Definition 2.3 [21] The *adelic Tamagawa measure* $\omega_{\mathcal{K},S}$ on $X(\mathbf{A}_K)$ is defined by

$$\omega_{\mathcal{K},S} = \prod_{v\in\mathrm{Val}(K)} L_v(1,T_{NS};E_{NS}/K)^{-1}\omega_{\mathcal{K},v}.$$

Definition 2.4 Let t be the rank of the group of K-rational characters $\hat{T}_K$ of T. Then the *Tamagawa number of* T is defined as

$$\tau(T) = \frac{b_S(T)}{l_S(T)}$$

where

$$b_S(T) = \int_{T^1(\mathbf{A}_K)/T(K)} \omega^1_{\Omega,S},$$

$$l_S(T) = \lim_{s\to 1}(s-1)^t L_S(s,T;E/K).$$

Definition 2.5 [21] Let k be the rank of the Néron-Severi group of X over K, and $\overline{X(K)}$ the closure of $X(K) \subset X(\mathbf{A}_K)$ in the direct product topology. Then the *Tamagawa number* of X is defined by

$$\tau_{\mathcal{K}}(X) = \frac{b_S(X)}{l_S(X)}$$

where

$$b_S(X) = \int_{\overline{X(K)}} \omega_{\mathcal{K},S}$$

whenever the adelic integral converges, and

$$l_S^{-1}(X) = \lim_{s \to 1} (s-1)^k L_S(s, T_{NS}; E_{NS}/K).$$

Remark 2.6 Notice the difference in the choice of convergence factors for the Tamagawa measure associated with a metrized canonical line bundle on a complete algebraic variety X and for the classical Tamagawa measure on an algebraic torus T. In the first case, we choose $L_v^{-1}(1, T_{NS}; E_{NS}/K)$ whereas in the second case one uses $L_v(1, T; E/K)$. This explains the difference in the definitions of $l_S(X)$ and $l_S(T)$.

Remark 2.7 For a toric variety $\mathbf{P}_\Sigma \supset T$ one can take $E_{NS} = E$, where E is a splitting field of T.

Remark 2.8 It is clear that in both definitions the Tamagawa numbers do not depend on the choice of the finite subset $S \subset \mathrm{Val}(K)$. E. Peyre ([21]) proves the existence of the Tamagawa number for Fano varieties by using the Weil conjectures. The same method shows the existence of the Tamagawa number for smooth complete varieties X satisfying the conditions $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$.

Theorem 2.9 [20] *Let T be an algebraic torus defined over K. The Tamagawa number $\tau(T)$ doesn't depend on the choice of a splitting field E/K. We have*

$$\tau(T) = h(T)/i(T).$$

The constants $h(T), i(T)$ were defined in 1.5.

We see that the Tamagawa number of an algebraic torus is a rational number. We have $\tau(\mathbf{G}_m(K)) = 1$. The Tamagawa number of a Fano variety with a metrized canonical line bundle is certainly not rational in general. For $\mathbf{P}^1_{\mathbf{Q}}$ with our metrization we have $\tau_{\mathcal{K}}(\mathbf{P}^1_{\mathbf{Q}}) = 1/\zeta_{\mathbf{Q}}(2)$.

Proposition 2.10 [2] *One has*

$$\int_{\overline{T(K)}} \omega_{\mathcal{K},S} = \int_{\overline{\mathbf{P}_{\Sigma}(K)}} \omega_{\mathcal{K},S}.$$

3 Heights and their Fourier transforms

Let $\varphi \in PL(\Sigma)^G_{\mathbf{C}}$. Using the decomposition of $\Sigma(1)$ into a union of G-orbits

$$\Sigma(1) = \Sigma_1(1) \cup ... \cup \Sigma_r(1),$$

we can identify φ with a T-invariant divisor with complex coefficients

$$D_{\varphi} = s_1 D_1 + \cdots + s_r D_r$$

where $s_j = \varphi(e_j) \in \mathbf{C}$ and e_j is a primitive lattice generator of some cone $\sigma \in \Sigma_j(1)$ $(j = 1, \ldots, r)$. It will be convenient to identify an element $\varphi = \varphi_{\mathbf{s}} \in PL(\Sigma)^G_{\mathbf{C}}$ with the vector $\mathbf{s} = (s_1, \ldots, s_r)$ of its complex coordinates.

Let us recall the definition of heights on toric varieties from [2]. For our purposes it will be sufficient to describe the restrictions of heights to the Zariski open subset $T \subset \mathbf{P}_{\Sigma,K}$.

Proposition 3.1 *Let $v \in \mathrm{Val}(K)$ be a valuation and $G_v \subset G$ the decomposition group of v. There is an injective homomorphism*

$$\pi_v : T(K_v)/T(\mathcal{O}_v) \hookrightarrow N_v,$$

which is an isomorphism for all but finitely many $v \in \mathrm{Val}(K)$. Here $N_v = N^{G_v} \subset N$ for non-archimedian v and $N_v = N^{G_v}_{\mathbf{R}}$ for archimedian valuations v. For every non-archimedian valuation we can identify the image of π_v with a sublattice of finite index in N_v.

Definition 3.2 Let $\mathbf{s} \in \mathbf{C}^r$ be a complex vector defining a complex piecewise linear G-invariant function $\varphi \in PL(\Sigma)^G_{\mathbf{C}}$. For any point $x_v \in T(K_v) \subset \mathbf{P}_\Sigma(K_v)$, denote by $\overline{x}_v$ the image of x_v in N_v, where N_v is considered as a canonical lattice in the real space $N^{G_v}_{\mathbf{R}}$ for non-archimedian valuations (resp. as the real Lie-algebra $N_{\mathbf{R},v}$ of $T(K_v)$ for archimedian valuations). Define the *complexified local Weil function* $H_{\Sigma,v}(x_v, \mathbf{s})$ by the formula

$$H_{\Sigma,v}(x_v, \mathbf{s}) = e^{\varphi(\overline{x}_v)\log q_v}$$

where q_v is the cardinality of the residue field k_v of K_v if v is non-archimedian and $\log q_v = 1$ if v is archimedian.

Theorem 3.3 [2] *The complexified local Weil function $H_{\Sigma,v}(x_v, \mathbf{s})$ satisfies the following properties:*

(i) *$H_{\Sigma,v}(x_v, \mathbf{s})$ is $T(\mathcal{O}_v)$-invariant.*

(ii) *If $\mathbf{s} = 0$, then $H_{\Sigma,v}(x_v, \mathbf{s}) = 1$ for all $x_v \in T(K_v)$.*

(iii) *$H_{\Sigma,v}(x_v, \mathbf{s} + \mathbf{s}') = H_{\Sigma,v}(x_v, \mathbf{s})H_{\Sigma,v}(x_v, \mathbf{s}')$.*

(iv) *If $\mathbf{s} = (s_1, ..., s_r) \in \mathbf{Z}^r$, then $H_{\Sigma,v}(x_v, \mathbf{s})$ is a classical local Weil function corresponding to a Cartier divisor $D_{\mathbf{s}} = s_1D_1 + \cdots + s_rD_r$ on $\mathbf{P}_{\Sigma,K}$.*

Definition 3.4 For a piecewise linear function $\varphi_{\mathbf{s}} \in PL(\Sigma)^G_{\mathbf{C}}$ we define the *complexified height function on* $T(K) \subset \mathbf{P}_{\Sigma,K}(K)$ by

$$H_\Sigma(x, \mathbf{s}) = \prod_{v \in \mathrm{Val}(K)} H_{\Sigma,v}(x_v, \mathbf{s}).$$

Remark 3.5 Although the local heights are defined only as functions on $PL(\Sigma)^G_{\mathbf{C}} \cong \mathbf{C}^r$, the product formula implies that for $x \in T(K)$ the global complexified height function descends to the Picard group $\mathrm{Pic}(\mathbf{P}_{\Sigma,K})_{\mathbf{C}}$. Moreover, since $H_\Sigma(x, \mathbf{s})$ is the product of local complex Weil functions $H_{\Sigma,v}(x, \mathbf{s})$ and since for all $x_v \in T(\mathcal{O}_v)$ we have $H_{\Sigma,v}(x_v, \mathbf{s}) = 1$ for all v, we can immediately extend $H_\Sigma(x, \mathbf{s})$ to a function on $T(\mathbf{A}_K) \times PL(\Sigma)^G_{\mathbf{C}}$.

Definition 3.6 Let $\Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_l(1)$ be the decomposition of $\Sigma(1)$ into a disjoint union of G_v-orbits. Denote by d_j the length of the G_v-orbit $\Sigma_j(1)$ ($d_1 + \cdots + d_l = n$). We establish a 1-to-1 correspondence $\Sigma_j(1) \leftrightarrow u_j$ between the G_v-orbits $\Sigma_1(1), \ldots, \Sigma_l(1)$ and independent variables $u_1, \ldots, u_l$.

Let $\sigma \in \Sigma^{G_v}$ be any G_v-invariant cone and $\Sigma_{j_1}(1) \cup \cdots \cup \Sigma_{j_k}(1)$ the set of all 1-dimensional faces of σ. We define the rational function $R_\sigma(u_1, \ldots, u_l)$ corresponding to σ as follows:

$$R_\sigma(u_1, \ldots, u_l) := \frac{u_{j_1}^{d_{j_1}} \cdots u_{j_k}^{d_{j_k}}}{(1 - u_{j_1}^{d_{j_1}}) \cdots (1 - u_{j_k}^{d_{j_k}})}.$$

Define the polynomial $Q_\Sigma(u_1, \ldots, u_l)$ by the formula

$$\sum_{\sigma \in \Sigma^{G_v}} R_\sigma(u_1, \ldots, u_l) = \frac{Q_\Sigma(u_1, \ldots, u_l)}{(1 - u_1^{d_1}) \cdots (1 - u_l^{d_l})}.$$

Proposition 3.7 [2] *Let Σ be a complete regular G_v-invariant fan. Then the polynomial*

$$Q_\Sigma(u_1, \ldots, u_l) - 1$$

contains only monomials of degree ≥ 2.

Let χ be a topological character of $T(\mathbf{A}_K)$ such that its v-component $\chi_v : T(K_v) \to S^1 \subset \mathbf{C}^*$ is trivial on $T(\mathcal{O}_v)$. For each $j \in \{1, \ldots, l\}$, we denote by n_j the sum of d_j generators of all 1-dimensional cones if the G_v-orbit $\Sigma_j(1)$. By (3.1) we know that the homomorphism

$$\pi_v : T(K_v)/T(\mathcal{O}_v) \to N_v$$

is an isomorphism for almost all v. We call these valuations *good*. Hence, for good non-archimedian valuations, n_j represents an element of $T(K_v)$ modulo $T(\mathcal{O}_v)$. Therefore, $\chi_v(n_j)$ is well-defined.

Definition 3.8 Denote by

$$\hat{H}_{\Sigma,v}(\chi_v, -\mathbf{s}) = \int_{T(K_v)} H_{\Sigma,v}(x_v, -\mathbf{s}) \chi_v(x_v) d\mu_v$$

the value at χ_v of the *multiplicative* Fourier transform of the local Weil function $H_{\Sigma,v}(x_v, -\mathbf{s})$ with respect to the v-adic Haar measure $d\mu_v$ on $T(K_v)$ normalized by $\int_{T(\mathcal{O}_v)} d\mu_v = 1$.

Proposition 3.9 [2] *Let v be a good non-archimedian valuation of K . For any topological character χ_v of $T(K_v)$ which is trivial on the subgroup $T(\mathcal{O}_v)$ and a piecewise linear function $\varphi = \varphi_{\mathbf{s}} \in PL(\Sigma)^G_{\mathbf{C}}$ one has*

$$\hat{H}_{\Sigma,v}(\chi_v, -\mathbf{s}) = \frac{Q_\Sigma\left(\frac{\chi_v(n_1)}{{q_v}^{\varphi(n_1)}}, \dots, \frac{\chi_v(n_l)}{{q_v}^{\varphi(n_l)}}\right)}{(1 - \frac{\chi_v(n_1)}{{q_v}^{\varphi(n_1)}}) \cdots (1 - \frac{\chi_v(n_l)}{{q_v}^{\varphi(n_l)}})}.$$

Corollary 3.10 [2] Let v be a good non-archimedian valuation of K. The restriction of

$$\int_{T(K_v)} H_{\Sigma,v}(x_v, -\mathbf{s}) d\mu_v$$

to the line $s_1 = \cdots = s_r = s$ is equal to

$$L_v(s, T; E/K) \cdot L_v(s, T_{NS}, E/K) \cdot Q_\Sigma(q_v^{-s}, \dots, q_v^{-s}).$$

Remark 3.11 It is difficult to calculate the Fourier transforms of local heights for the finitely many "bad" non-archimedian valuations v, because there is only an embedding of finite index

$$T(K_v)/T(\mathcal{O}_v) \hookrightarrow N_v.$$

However, for our purposes it will be sufficient to use upper estimates for these local Fourier transforms. One immediately sees that for all non-archimedian valuations v the local Fourier transforms of $H_{\Sigma,v}(x_v, -\mathbf{s})$ can be bounded absolutely and uniformly in all characters by a finite combination of multi-dimensional geometric series in $q_v^{-1/2}$ in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}_{>1/2}$.

Now we assume that v is an archimedian valuation. By (3.1), we have $T(K_v)/T(\mathcal{O}_v) = N_{\mathbf{R}}^{G_v} \subset N_{\mathbf{R}}$ where G_v is the trivial group for the case $K_v = \mathbf{C}$, and $G_v = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \cong \mathbf{Z}/2\mathbf{Z}$ for the case $K_v = \mathbf{R}$. Let $\langle \cdot, \cdot \rangle$ be the pairing between $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ induced from the duality between N and M. Let y be an arbitrary element of the dual $\mathbf{R}$-space $M_{\mathbf{R}}^{G_v} = \mathrm{Hom}(T(K_v)/T(\mathcal{O}_v), \mathbf{R})$. Then $\chi_y(x_v) = e^{-i\langle \overline{x}_v, y \rangle}$ is a topological character of $T(K_v)$ which is trivial on $T(\mathcal{O}_v)$. We choose the Haar measure $d\mu_v$ on $T(K_v)$ as the product of the Haar measure $d\mu_v^0$ on $T(\mathcal{O}_v)$ and the Haar measure $d\overline{x}_v$ on $T(K_v)/T(\mathcal{O}_v)$. We normalize the measures such that the $d\mu_v^0$-volume of $T(\mathcal{O}_v)$ equals 1 and $d\overline{x}_v$ is the standard Lebesgue measure on $N_{\mathbf{R}}^{G_v}$ normalized by the full sublattice N^{G_v}.

Proposition 3.12 [2] *Let v be an archimedean valuation of K. The Fourier transform $\hat{H}_{\Sigma,v}(\chi_y, -\mathbf{s})$ of a local archimedean Weil function*

$$H_{\Sigma,v}(x_v, -\mathbf{s}) = e^{-\varphi_{\mathbf{s}}(\overline{x}_v)}$$

is a rational function in variables $s_j = \varphi_{\mathbf{s}}(e_j)$ for $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}_{>0}$.

Proof. Let us consider the case $K_v = \mathbf{C}$. One has a decomposition of the space $N_{\mathbf{R}}$ into a union of d-dimensional cones $N_{\mathbf{R}} = \bigcup_{\sigma \in \Sigma(d)} \sigma$. We calculate the Fourier transform as follows:

$$\hat{H}_{\Sigma,v}(\chi_y, -\mathbf{s}) = \int_{N_{\mathbf{R}}} e^{-\varphi_{\mathbf{s}}(\overline{x}_v) - i\langle \overline{x}_v, y\rangle} d\overline{x}_v =$$

$$= \sum_{\sigma \in \Sigma(d)} \int_{\sigma} e^{-\varphi_{\mathbf{s}}(\overline{x}_v) - i\langle \overline{x}_v, y\rangle} d\overline{x}_v = \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_j \in \sigma}(s_j + i\langle e_j, y\rangle)}.$$

The case $K_v = \mathbf{R}$ can be reduced to the above situation. □

4 Poisson formula

Let $\mathbf{P}_\Sigma$ be a toric variety and $H_\Sigma(x, \mathbf{s})$ the height function constructed above.

Definition 4.1 We define the zeta-function of the complex height-function $H_\Sigma(x, \mathbf{s})$ as

$$Z_\Sigma(\mathbf{s}) = \sum_{x \in T(K)} H_\Sigma(x, -\mathbf{s}).$$

Theorem 4.2 *The series $Z_\Sigma(\mathbf{s})$ converges absolutely and uniformly for* $\mathbf{s}$ *contained in any compact in the domain* $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$.

Proof. It was proved in [19] that we can always choose a finite set S such that the natural map

$$\pi_S \; : \; T(K) \to \bigoplus_{v \notin S} T(K_v)/T(\mathcal{O}_v) = \bigoplus_{v \notin S} N_v$$

is surjective. Denote by $T(\mathcal{O}_S)$ the kernel of π_S consisting of all S-units in $T(K)$. Let $W(T) \subset T(\mathcal{O}_S)$ the subgroup of torsion elements in $T(\mathcal{O}_S)$. Then

$T(\mathcal{O}_S)/W(T)$ has a natural embedding into the finite-dimensional logarithmic space

$$N_{\mathbf{R},S} = \bigoplus_{v \in S} T(K_v)/T(\mathcal{O}_v) \otimes \mathbf{R}$$

as a sublattice of codimension t. Let Γ be a full sublattice in $N_{\mathbf{R},S}$ containing the image of $T(\mathcal{O}_S)/W(T)$. Denote by Δ a bounded fundamental domain of Γ in $N_{\mathbf{R},S}$. For any $x \in T(K)$ we denote by $\overline{x}_S$ the image of x in $N_{\mathbf{R},S}$. Define $\phi(x)$ to be the element of Γ such that $\overline{x}_S - \phi(x) \in \Delta$. Thus, we have obtained the mapping

$$\phi \; : \; T(K) \rightarrow \Gamma.$$

Define a new function $\tilde{H}_\Sigma(x, \mathbf{s})$ on $T(K)$ by

$$\tilde{H}_\Sigma(x, \mathbf{s}) = \prod_{v \in S} H_{\Sigma,v}(\phi(x)_v, \mathbf{s}) \prod_{v \notin S} H_{\Sigma,v}(x_v, \mathbf{s}).$$

If $\mathbf{K} \subset \mathbf{C}^r$ is a compact in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$, then there exist two positive constants $C_1(\mathbf{K}) < C_2(\mathbf{K})$ such that

$$0 < C_1(\mathbf{K}) < \frac{\tilde{H}_\Sigma(x, \mathbf{s})}{H_\Sigma(x, \mathbf{s})} < C_2(\mathbf{K}) \;\text{ for } \mathbf{s} \in \mathbf{K},\, x \in T(K),$$

since $\overline{x}_v - \phi(x)_v$ belongs to some bounded subset Δ_v in $N_{\mathbf{R},v}$ for any $x \in T(K)$ and $v \in S$. Therefore, it is sufficient to prove that the series

$$\tilde{Z}_\Sigma(\mathbf{s}) = \sum_{x \in T(K)} \tilde{H}_\Sigma(x, -\mathbf{s})$$

is absolutely converent for $\mathbf{s} \in \mathbf{K}$. Notice that $\tilde{Z}_\Sigma(\mathbf{s})$ can be estimated from above by the the following Euler product

$$\left(\sum_{\gamma \in \Gamma} \prod_{v \in S} H_{\Sigma,v}(\gamma_v, -\mathbf{s}) \right) \prod_{v \notin S} \left(\sum_{z \in N_v} H_{\Sigma,v}(z, -\mathbf{s}) \right).$$

The sum

$$\sum_{\gamma \in \Gamma} \prod_{v \in S} H_{\Sigma,v}(\gamma_v, -\mathbf{s})$$

is an absolutely convergent geometric series for $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$. On the other hand, the Euler product

$$\prod_{v \notin S} \left(\sum_{z \in N_v} H_{\Sigma,v}(z, -\mathbf{s}) \right)$$

can be estimated from above by the product of zeta-functions

$$C_3(\mathbf{K}) \prod_{j=1}^{r} \zeta_{K_j}(s_j),$$

where $C_3(\mathbf{K})$ is some constant depending on $\mathbf{K}$. Since each $\zeta_{K_j}(s_j)$ is absolutely convergent for $\mathrm{Re}(s_j) > 1$, we obtain the statement. □

We need the Poisson formula in the following form:

Theorem 4.3 ([11], 31.46 e) *Let $\mathcal{G}$ be a locally compact abelian group with Haar measure $dg, \mathcal{H} \subset \mathcal{G}$ a closed subgroup with Haar measure dh. The factor group $\mathcal{G}/\mathcal{H}$ has a unique Haar measure dx normalized by the condition $dg = dx \cdot dh$. Let $F : \mathcal{G} \to \mathbf{R}$ be an L^1-function on $\mathcal{G}$ and $\hat{F}$ its Fourier transform with respect to dg. Suppose that $\hat{F}$ is also an L^1-function on $\mathcal{H}^\perp$, where $\mathcal{H}^\perp$ is the group of topological characters $\chi : \mathcal{G} \to S^1$ which are trivial on $\mathcal{H}$. Then*

$$\int_{\mathcal{H}} F(x) dh = \int_{\mathcal{H}^\perp} \hat{F}(\chi) d\chi,$$

where $d\chi$ is the orthogonal Haar measure on $\mathcal{H}^\perp$ with respect to the Haar measure dx on $\mathcal{G}/\mathcal{H}$.

We will apply this theorem in the case when $\mathcal{G} = T(\mathbf{A}_K)$ and $\mathcal{H} = T(K)$, $dg = \omega_{\Omega,S}$, and dh is the discrete measure on $T(K)$.

Theorem 4.4 *(Poisson formula) For all $\mathbf{s}$ with $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$ we have the following formula:*

$$Z_\Sigma(\mathbf{s}) = \frac{1}{(2\pi)^t b_S(T)} \int_{(T(\mathbf{A}_K)/T(K))^*} \left(\int_{T(A_F)} H_\Sigma(x, -\mathbf{s}) \chi(x) \omega_{\Omega,S} \right) d\chi,$$

where $\chi \in (T(\mathbf{A}_K)/T(K))^*$ is a topological character of $T(\mathbf{A}_K)$, trivial on the closed subgroup $T(K)$ and $d\chi$ is the orthogonal Haar measure on $(T(\mathbf{A}_K)/T(K))^*$. The integral converges absolutely and uniformly to a holomorphic function in $\mathbf{s}$ in any compact in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$.

Proof. Because of 4.2 we only need to show that the Fourier transform $\hat{H}_\Sigma(\chi, -\mathbf{s})$ of the height function is an L^1-function on $(T(\mathbf{A}_K)/T(K))^*$. By

3.9 and uniform estimates at places of bad reduction 3.11, we know that the Euler product

$$\prod_{v \notin S_\infty} \hat{H}_{\Sigma,v}(\chi_v, -\mathbf{s})$$

converges absolutely and is uniformly bounded by a constant $c(\mathbf{K})$ for all characters χ and all $\mathbf{s} \in \mathbf{K}$, where $\mathbf{K}$ is some compact in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$.

Since the height function $H_{\Sigma,v}(x, -\mathbf{s})$ is invariant under $T(\mathcal{O}_v)$ for all v, the Fourier transform of $\hat{H}_\Sigma(\chi, -\mathbf{s})$ equals zero for characters χ which are non-trivial on the maximal compact subgroup $\mathbf{K}_T$. Denote by $\mathcal{P}$ the set of all such characters $\chi \in (T(\mathbf{A}_K)/T(K))^*$.

We have a non-canonical splitting of characters $\chi = \chi_l \cdot \chi_y$, where $\chi_l \in (T^1(\mathbf{A}_K)/T(K))^*$ and $\chi_y \in (T(\mathbf{A}_K)/T^1(\mathbf{A}_K))^*$. Let us consider the logarithmic space

$$N_{\mathbf{R},\infty} = \bigoplus_{v \in S_\infty} T(K_v)/T(\mathcal{O}_v) = \bigoplus_{v \in S_\infty} N_{\mathbf{R},v}.$$

It contains the lattice $T(\mathcal{O}_K)/W(T)$ of $\mathcal{O}_K$-integral points of $T(K)$ modulo torsion. Denote by $M_{\mathbf{R},\infty} = \bigoplus_{v \in S_\infty} M_{\mathbf{R},v}$ the dual space. It has a decomposition as a direct sum of vector spaces $M_{\mathbf{R},\infty} = M_L \oplus M_Y$, such that the space M_L contains the dual lattice $L := (T(\mathcal{O}_K)/W(T))^*$ as a full sublattice and the space M_Y is isomorphic to $(T(\mathbf{A}_K)/T^1(\mathbf{A}_K))^* = \hat{T}_K \otimes \mathbf{R}$.

By 1.4, we have an exact sequence

$$0 \to \mathbf{cl}^*(T) \to \mathcal{P} \to \mathcal{M} \to 0,$$

where $\mathcal{M}$ is the image of the projection of $\mathcal{P}$ to $M_{\mathbf{R},\infty}$ and $\mathbf{cl}^*(T)$ is a finite group. We see that the character $\chi \in \mathcal{P}$ is determined by its archimedian component up to a finite choice. Denote by $y(\chi) \in \mathcal{M} \subset M_{\mathbf{R},\infty}$ the image of $\chi \in \mathcal{P}$.

The following lemmas will provide the necessary estimates of the Fourier transform of local heights at archimedian places. This allows to apply the Poisson formula 4.3. □

Lemma 4.5 [2] Let $\Sigma \subset N_{\mathbf{R}}$ be a complete fan in a real vector space of dimension d. Denote by $M_{\mathbf{R}}$ the dual space. For all $m \in M_{\mathbf{R}}$ and all $\mathbf{s}$ with $0 < \delta_1 < \mathrm{Re}(s_j) < \delta_2$ there exists a constant $c = c(\delta_1, \delta_2, \Sigma)$ such that we have the following estimate

$$|\sum_{\sigma\in\Sigma(d)} \frac{1}{\prod_{e_j\in\sigma}(s_j + i < e_j, m >)}| \le c \sum_{\sigma\in\Sigma(d)} \frac{1}{\prod_{e_j\in\sigma}(1 + | < e_j, m > |)^{1+1/d}}.$$

Corollary 4.6 Consider

$$\hat{H}_{\Sigma,\infty}(\chi, -\mathbf{s}) = \prod_{v\in S_\infty} \hat{H}_{\Sigma,v}(\chi, -\mathbf{s})$$

as a function on

$$\mathcal{M} \subset M_{\mathbf{R},\infty} = \bigoplus_{v\in S_\infty} M_{\mathbf{R},v}.$$

Let d' be the dimension of $M_{\mathbf{R},\infty}$. We have a direct sum decomposition $M_{\mathbf{R},\infty} = M_L \oplus M_Y$ of real vector spaces. Let $M'_Y \subset M_Y$ be any affine subspace, dy' the Lebesgue measure on M'_Y and $L' \subset M_L$ any lattice. Let $g(y, -\mathbf{s})$ be a function on $M_{\mathbf{R},\infty}$ satisfying $|g(y, -\mathbf{s})| \le c(1 + \|y\|)^\delta$ for all $y \in M_{\mathbf{R},\infty}$, all $\mathbf{s}$ contained in some compact $\mathbf{K} \subset \mathbf{C}^r$ in the domain $\mathbf{R}^r_{>1/2}$, some $0 < \delta < 1/d'$ and some constant $c > 0$. Then the series

$$\sum_{y(\chi)\in L} \int_{M'_Y} g(y, -\mathbf{s})\hat{H}_{\Sigma,\infty}(y(\chi), -\mathbf{s})dy'$$

is absolutely and uniformly convergent to a holomorphic function in $\mathbf{s}$ in this domain.

Proof. We apply 3.12 and observe that on the space $N_{\mathbf{R},\infty}$ we have a fan Σ_∞ obtained as the direct product of fans Σ^{G_v} for $v \in S_\infty$ (i.e., every cone in Σ_∞ is a direct product of cones in Σ^{G_v}). □

5 $\mathcal{X}$-functions of convex cones

Let $(A, A_{\mathbf{R}}, \Lambda)$ be a triple consisting of a free abelian group A of rank k, a k-dimensional real vector space $A_{\mathbf{R}} = A \otimes \mathbf{R}$ containing A as a sublattice of maximal rank, and a convex k-dimensional cone $\Lambda \subset A_{\mathbf{R}}$ such that $\Lambda \cap -\Lambda = 0 \in A_{\mathbf{R}}$. Denote by Λ° the interior of Λ and by $\Lambda^\circ_{\mathbf{C}} = \Lambda^\circ + iA_{\mathbf{R}}$ the complex tube domain over Λ°. Let $(A^*, A^*_{\mathbf{R}}, \Lambda^*)$ be the triple consisting of the dual abelian group $A^* = \mathrm{Hom}(A, \mathbf{Z})$, the dual real vector space $A^*_{\mathbf{R}} = \mathrm{Hom}(A_{\mathbf{R}}, \mathbf{R})$, and the dual cone $\Lambda^* \subset A^*_{\mathbf{R}}$. We normalize the Haar measure $\mathbf{dy}$ on $A^*_{\mathbf{R}}$ by the condition: $\mathrm{vol}(A^*_{\mathbf{R}}/A^*) = 1$.

Definition 5.1 The *$\mathcal{X}$-function of* Λ is defined as the integral

$$\mathcal{X}_\Lambda(\mathbf{s}) = \int_{\Lambda^*} e^{-\langle \mathbf{s},\mathbf{y}\rangle} \mathbf{dy},$$

where $\mathbf{s} \in \Lambda^\circ_{\mathbf{C}}$.

Remark 5.2 $\mathcal{X}$-functions of convex cones have been investigated in the theory of homogeneous cones by M. Köcher, O.S. Rothaus, and E.B. Vinberg [12, 23, 25]. In these papers $\mathcal{X}$-functions were called *characteristic functions of cones*, but we find such a notion rather misleading in view of the fact that $\mathcal{X}_\Lambda(\mathbf{s})$ is the Fourier-Laplace transform of the standard set-theoretic characteristic function of the dual cone Λ^*.

The function $\mathcal{X}_\Lambda(\mathbf{s})$ has the following properties [23, 25]:

Proposition 5.3 (i) *If* $\mathcal{A}$ *is any invertible linear operator on* $\mathbf{R}^k$*, then*

$$\mathcal{X}_\Lambda(\mathcal{A}\mathbf{s}) = \frac{\mathcal{X}_\Lambda(\mathbf{s})}{\det \mathcal{A}};$$

(ii) *If* $\Lambda = \mathbf{R}^k_{\geq 0}$*, then*

$$\mathcal{X}_\Lambda(\mathbf{s}) = (s_1 \cdots s_k)^{-1}, \text{ for } \mathrm{Re}(s_i) > 0;$$

(iii) *If* $\mathbf{s} \in \Lambda^\circ$*, then*

$$\lim_{\mathbf{s} \to \partial\Lambda} \mathcal{X}_\Lambda(\mathbf{s}) = \infty;$$

(iv) $\mathcal{X}_\Lambda(\mathbf{s}) \neq 0$ *for all* $\mathbf{s} \in \Lambda^\circ_{\mathbf{C}}$*.*

Proposition 5.4 *If* Λ *is a* k*-dimensional finitely generated polyhedral cone, then* $\mathcal{X}_\Lambda(\mathbf{s})$ *is a rational function*

$$\mathcal{X}_\Lambda(\mathbf{s}) = \frac{P(\mathbf{s})}{Q(\mathbf{s})},$$

where P *is a homogeneous polynomial,* Q *is a product of all linear homogeneous forms defining the codimension* 1 *faces of* Λ*, and* $\deg P - \deg Q = -k$*.*

Proof. We subdivide the dual cone Λ^* into a finite union of simplicial subcones Λ_j^* $(j \in J)$. Let $\Lambda_j \subset A_{\mathbf{R}}$ be the dual cone to Λ_j^*. Then

$$\mathcal{X}_\Lambda(\mathbf{s}) = \sum_{j \in J} \mathcal{X}_{\Lambda_j}(\mathbf{s}).$$

By 5.3(i) and (ii),

$$\mathcal{X}_{\Lambda_j}(\mathbf{s}) = \frac{P_j(\mathbf{s})}{Q_j(\mathbf{s})} \ \ (j \in J),$$

where P_j is a homogeneous polynomial of degree 0 and Q_j is the product of k homogeneous linear forms defining the codimension 1 faces of Λ_j. Therefore, $\mathcal{X}_\Lambda(\mathbf{s})$ can be uniquely represented up to constants as a ratio of two homogeneous polynomials $P(\mathbf{s})/Q(\mathbf{s})$ with $g.c.d.(P, Q) = 1$ where Q is a product of linear homogeneous forms defining some faces of Λ_j of codimension 1. Since Q does not depend on a choice of a subdivision of Λ^* into a finite union of simplicial cones Λ_j^*, only linear homogeneous forms which vanish on codimension 1 faces of Λ can be factors of Q. On the other hand, by 5.3(iii), every linear homogeneous form vanishing on a face of Λ of codimension 1 must divide Q. □

Theorem 5.5 *Let* $(A, A_{\mathbf{R}}, \Lambda)$ *and* $(\tilde{A}, \tilde{A}_{\mathbf{R}}, \tilde{\Lambda})$ *be two triples as above,* $k = \operatorname{rk} A$ *and* $\tilde{k} = \operatorname{rk} \tilde{A}$*, and* $\psi : A \to \tilde{A}$ *a homomorphism of free abelian groups with a finite cokernel* A' *(i.e., the corresponding linear mapping of real vector spaces* $\psi : A_{\mathbf{R}} \to \tilde{A}_{\mathbf{R}}$ *is surjective), and* $\psi(\Lambda) = \tilde{\Lambda}$*. Let* $B = \operatorname{Ker} \psi \subset A$*,* $\mathbf{db}$ *the Haar measure on* $B_{\mathbf{R}} = B \otimes \mathbf{R}$ *normalized by the condition* $\operatorname{vol}(B_{\mathbf{R}}/B) = 1$*. Then for all* $\mathbf{s}$ *with* $\operatorname{Re}(\mathbf{s}) \in \Lambda^\circ$ *the following formula holds:*

$$\mathcal{X}_{\tilde{\Lambda}}(\psi(\mathbf{s})) = \frac{1}{(2\pi)^{k-\tilde{k}}|A'|} \int_{B_{\mathbf{R}}} \mathcal{X}_\Lambda(\mathbf{s} + i\mathbf{b})\mathbf{db},$$

where $|A'|$ *is the order of the finite abelian group* A'*.*

Proof. We have the dual injective homomorphisms of free abelian groups $\psi^* : \tilde{A}^* \to A^*$ and of the corresponding real vector spaces $\psi^* : \tilde{A}^*_{\mathbf{R}} \to A^*_{\mathbf{R}}$. Moreover, $\tilde{\Lambda}^* = \Lambda^* \cap \tilde{A}^*_{\mathbf{R}}$. Let $C_{\Lambda^*}(\mathbf{y})$ be the set-theoretic characteristic function of the cone $\Lambda^* \subset A^*_{\mathbf{R}}$ and $C_{\Lambda^*}(\tilde{\mathbf{y}})$ the restriction of $C_{\Lambda^*}(\mathbf{y})$ to $\tilde{A}^*_{\mathbf{R}}$ which is the set-theoretic characteristic function of $\tilde{\Lambda}^* \subset \tilde{A}^*_{\mathbf{R}}$. Then

$$\mathcal{X}_{\tilde{\Lambda}}(\psi(\mathbf{s})) = \int_{\tilde{A}^*_{\mathbf{R}}} C_{\Lambda^*}(\tilde{\mathbf{y}}) e^{-\langle \psi(\mathbf{s}), \tilde{\mathbf{y}} \rangle} \mathbf{d}\tilde{\mathbf{y}}.$$

Now we apply the Poisson formula to the last integral. For this purpose we notice that any additive topological character of $A^*_{\mathbf{R}}$ which vanishes on the subgroup $\tilde{A}^*_{\mathbf{R}} \subset A^*_{\mathbf{R}}$ has the form

$$e^{-i\langle \mathbf{b}, \mathbf{y}\rangle}, \quad \text{where } \mathbf{b} \in B_{\mathbf{R}}.$$

Moreover,

$$\frac{\mathbf{db}}{(2\pi)^{k-\tilde{k}}|A'|}$$

is the orthogonal Haar measure on $B_{\mathbf{R}}$ with respect to the Haar measures $\mathbf{d\tilde{y}}$ and $\mathbf{dy}$ on $\tilde{A}^*_{\mathbf{R}}$ and $A^*_{\mathbf{R}}$ respectively. It remains to notice that

$$\mathcal{X}_{\Lambda}(\mathbf{s} + i\mathbf{b}) = \int_{A^*_{\mathbf{R}}} C_{\Lambda^*}(\mathbf{y}) e^{-\langle \mathbf{s}+i\mathbf{b}, \mathbf{y}\rangle} \mathbf{dy}$$

is the value of the Fourier transform of $C_{\Lambda^*}(\mathbf{y})e^{-\langle \mathbf{s}, \mathbf{y}\rangle}$ on the topological character of $A^*_{\mathbf{R}}/\tilde{A}^*_{\mathbf{R}}$ corresponding to an element $\mathbf{b} \in B_{\mathbf{R}} \subset A_{\mathbf{R}}$. □

Corollary 5.6 Assume that in the above situation rk $= k - \tilde{k} = 1$ and $\tilde{A} = A/B$. Denote by γ a generator of B. Then

$$\mathcal{X}_{\tilde{\Lambda}}(\psi(\mathbf{s})) = \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=0} \mathcal{X}_{\Lambda}(\mathbf{s} + z \cdot \gamma) dz,$$

where $z = x + iy \in \mathbf{C}$.

Corollary 5.7 Assume that a $\tilde{k}$-dimensional rational finite polyhedral cone $\tilde{\Lambda} \subset \tilde{A}_{\mathbf{R}}$ contains exactly r one-dimensional faces with primitive lattice generators $a_1, \ldots, a_r \in \tilde{A}$. We set $k := r$, $A := \mathbf{Z}^r$ and denote by ψ the natural homomorphism of lattices $\mathbf{Z}^r \to \tilde{A}$ which sends the standard basis of $\mathbf{Z}^r$ into $a_1, \ldots, a_r \in \tilde{A}$, so that $\tilde{\Lambda}$ is the image of the simplicial cone $\mathbf{R}^r_{\geq 0} \subset \mathbf{R}^r$ under the surjective map of vector spaces $\psi : \mathbf{R}^r \to A_{\mathbf{R}}$. Denote by $M_{\mathbf{R}}$ the kernel of ψ and set $M := \mathbf{Z}^r \cap M_{\mathbf{R}}$. Let $\mathbf{s} = (s_1, ..., s_r)$ be the standard coordinates in $\mathbf{C}^r$. Then

$$\mathcal{X}_{\Lambda}(\psi(\mathbf{s})) = \frac{1}{(2\pi)^{r-k}|A'|} \int_{M_{\mathbf{R}}} \frac{1}{\prod_{j=1,n}(s_j + iy_j)} \mathbf{dy}$$

where $\mathbf{dy}$ is the Haar measure on the additive group $M_{\mathbf{R}}$ normalized by the lattice M, y_j are the coordinates of $\mathbf{y}$ in $\mathbf{R}^r$, and $|A'|$ is the index of the sublattice in $\tilde{A}$ generated by $a_1, \ldots, a_r$.

Example 5.8 Consider an example of a non-simplicial convex cone which appears as the cone of effective divisors of the split toric Del Pezzo surface X of anticanonical degree 6. The cone Λ_{eff} has 6 generators corresponding to exceptional curves of the first kind on X. We can construct X as the blow up of 3 points p_1, p_2, p_3 in general position in $\mathbf{P}^2$. Denote the exceptional curves by $C_1, C_2, C_3, C_{12}, C_{13}, C_{23}$, where C_{ij} is the proper pullback of the line joining p_i and p_j. Let $\mathbf{s} = s_1[C_1]+s_2[C_2]+s_3[C_3]+s_{12}[C_{12}]+s_{13}[C_{13}]+s_{23}[C_{23}] \in \Lambda_{\mathrm{eff}}^{\circ}$ be an element in the interior of the cone of effective divisors. The sublattice $M \subset \mathbf{Z}^6$ of rank 2 consisting of principal divisors is generated by $\gamma_1 = C_1 + C_{13} - C_2 - C_{23}$ and $\gamma_2 = C_1 + C_{12} - C_3 - C_{23} = 0$.

In our case, the integral formula in 5.7 is a 2-dimensional integral ($r = 6$) which can be computed by applying twice the residue theorem to two 1-dimensional integrals like the one in 5.6. We obtain the following formula for the characteristic function of Λ_{eff}:

$$\mathcal{X}_{\Lambda}(\psi(\mathbf{s})) = \frac{s_1 + s_2 + s_3 + s_{12} + s_{13} + s_{23}}{(s_1 + s_{23})(s_2 + s_{13})(s_3 + s_{12})(s_1 + s_2 + s_3)(s_{12} + s_{13} + s_{23})}.$$

Definition 5.9 Let X be a smooth proper algebraic variety. Consider the triple $(\mathrm{Pic}(X), \mathrm{Pic}(X) \otimes \mathbf{R}, \Lambda_{\mathrm{eff}})$ where $\Lambda_{\mathrm{eff}} \subset \mathrm{Pic}(X) \otimes \mathbf{R}$ is the cone generated by classes of effective divisors on X. Assume that the anticanonical class $[\mathcal{K}^{-1}] \in \mathrm{Pic}(X)_{\mathbf{R}}$ is contained in the interior of Λ_{eff}. We define the constant $\alpha(X)$ by

$$\alpha(X) = \mathcal{X}_{\Lambda_{\mathrm{eff}}}([\mathcal{K}^{-1}]).$$

Corollary 5.10 If Λ_{eff} is a finitely generated rational polyhedral cone, then $\alpha(X)$ is a rational number.

6 Some technical statements

Let E be a number field and χ an unramified Hecke character on $\mathbf{G}_m(A_E)$. Its local components χ_v for all valuations v are given by:

$$\chi_v : \mathbf{G}_m(E_v)/\mathbf{G}_m(\mathcal{O}_v) \to S^1$$

$$\chi_v(x_v) = |x_v|_v^{it_v}.$$

Definition 6.1 Let χ be an unramified Hecke character. We set

$$y(\chi) := \{t_v\}_{v\in S_\infty(E)} \in \mathbf{R}^{r_1+r_2},$$

where r_1 (resp. r_2) is the number of real (resp. pairs of complex) valuations of E. We also set

$$\|y(\chi)\| := \max_{v\in S_\infty(E)} |t_v|.$$

We will need uniform estimates for Hecke L-functions in vertical strips. They can be deduced using the Phragmen-Lindelöf principle [22].

Theorem 6.2 *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $0 < \delta_1 < \delta$ there exists a constant $c(\varepsilon, \delta_1, E) > 0$ such that the inequality*

$$|L_E(s,\chi)| \leq c(\varepsilon, \delta_1, E)(1 + |\mathrm{Im}(s)| + \|y(\chi)\|)^\varepsilon$$

holds for all s with $\delta_1 < |\mathrm{Re}(s) - 1| < \delta$ and every Hecke L-function $L_E(s,\chi)$ corresponding to an unramified Hecke character χ.

Corollary 6.3 For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any compact $\mathbf{K} \subset \mathbf{C}$ in the domain $0 < |\mathrm{Re}\,(s) - 1| < \delta$ there exists a constant $C(\mathbf{K}, \varepsilon, E)$ depending on $\mathbf{K}$, ε and E such that

$$|L_E(s,\chi)| \leq C(\mathbf{K}, \varepsilon, E)(1 + \|y(\chi)\|)^\varepsilon$$

for $s \in \mathbf{K}$ and every unramified character χ.

Let Σ be the Galois-invariant fan defining $\mathbf{P}_\Sigma$ and $\Sigma(1) = \Sigma_1(1) \cup ... \cup \Sigma_r(1)$ the decomposition of the set of one-dimensional generators of Σ into G-orbits. Let e_j be a primitive integral generator of σ_j, $G_j \subset G$ the stabilizer of e_j. Denote by $K_j \subset E$ the subfield of G_j-fixed elements. Consider the n-dimensional torus

$$T' = \prod_{j=1}^{r} R_{K_j/K}(\mathbf{G}_m).$$

Let us recall the exact sequence of Galois-modules from Proposition 1.15:

$$0 \to M^G \to PL(\Sigma)^G \to \mathrm{Pic}(\mathbf{P}_\Sigma) \to H^1(G, M) \to 0.$$

It induces a map of tori $T' \to T$ and a homomorphism

$$a \,:\, T'(\mathbf{A}_K)/T'(K) \to T(\mathbf{A}_K)/T(K).$$

So we get a dual homomorphism for characters

$$a^* : (T(\mathbf{A}_K)/T(K))^* \to \prod_{j=1}^{r} (\mathbf{G}_m(\mathbf{A}_{K_j})/\mathrm{G}_m(K_j))^*.$$

Proposition 6.4 [7] *The kernel of a^* is dual to the obstruction group to weak approximation $A(T)$ defined in 1.6.*

Let $\chi \in (T(\mathbf{A}_K)/T(K))^*$ be a character. Then $\chi \circ a$ defines r Hecke characters of the idele groups

$$\chi_j \,:\, \mathbf{G}_m(\mathbf{A}_{K_j}) \to S^1 \subset \mathbf{C}^*,\ j = 1, \dots, r.$$

If χ is trivial on $\mathbf{K}_T$, then all characters χ_j $(j = 1, \dots, r)$ are trivial on the maximal compact subgroups in $\mathbf{G}_m(\mathbf{A}_{K_j})$. We denote by $L_{K_j}(s, \chi_j)$ the Hecke L-function corresponding to the unramified character χ_j.

Proposition 6.5 *Let $\chi = (\chi_v)$ be a character and $\hat{H}_{\Sigma,v}(\chi_v, -\mathbf{s})$ the local Fourier transform of the complex local height function $H_{\Sigma,v}(x_v, -\mathbf{s})$. For any compact $\mathbf{K} \subset \mathbf{C}^r$ contained in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1/2}$ there exists a constant $c(\mathbf{K})$ such that*

$$\prod_{v \notin S} \hat{H}_{\Sigma,v}(\chi_v, -\mathbf{s}) \cdot \prod_{i=1}^{r} L^{-1}_{K_j}(s_j, \chi_j) \leq c(\mathbf{K})$$

for all characters $\chi \in (T(\mathbf{A}_K)/T(K))^*$.

The proof follows from explicit computations of local Fourier transforms 3.9 and is almost identical with the proof of Proposition 3.1.3 in [2].

Proposition 6.6 *There exists an $\varepsilon > 0$ such that for any open $U \subset \mathbf{C}^r$ contained in the domain $0 < |\mathrm{Re}(s_j) - 1| < \varepsilon$ for $j = 1, ..., r$ the integral*

$$\int_{(T(\mathbf{A}_K)/T(K))^*} \hat{H}_\Sigma(\chi, -\mathbf{s}) d\chi$$

converges absolutely and uniformly to a holomorphic function for $\mathrm{Re}(\mathbf{s}) \in U$.

Proof. Using uniform estimates of Fourier transforms for non-archimedian places of bad reduction (3.11) and the proposition above we need only to consider the following integral

$$\int_{(T(\mathbf{A}_K)/T(K))^*} \hat{H}_{\Sigma,\infty}(\chi, -\mathbf{s}) \prod_{j=1}^{r} L_{K_j}(s_j, \chi_j) d\chi.$$

Observe that there exist constants $c_1 > 0$ and $c_2 > 0$ such that we have the following inequalities:

$$c_1 \|y(\chi)\| \leq \sum_{j=1}^{r} \|y(\chi_j)\| \leq c_2 \|y(\chi)\|.$$

Here we denoted by $\|y(\chi)\|$ the norm of $y(\chi) \in M_{\mathbf{R},\infty}$. Recall that since we only consider χ which are trivial on the maximal compact subgroup $\mathbf{K}_T$, all characters χ_j are unramified. To conclude, we apply uniform estimates of Hecke L-functions from Corollary 6.3 and the Corollary 4.6. □

The rest of this section is devoted to the proof of our main technical result.

Let $\mathbf{R}[\mathbf{s}]$ (resp. $\mathbf{C}[\mathbf{s}]$) be the ring of polynomials in $s_1, \ldots, s_r$ with coefficients in $\mathbf{R}$ (resp. in $\mathbf{C}$), $\mathbf{C}[[\mathbf{s}]]$ the ring of formal power series in $s_1, \ldots, s_r$ with complex coefficients.

Definition 6.7 Two elements $f(\mathbf{s}), g(\mathbf{s}) \in \mathbf{C}[[\mathbf{s}]]$ will be called *coprime*, if *g.c.d.*$(f(\mathbf{s}), g(\mathbf{s})) = 1$.

Definition 6.8 Let $f(\mathbf{s})$ be an element of $\mathbf{C}[[\mathbf{s}]]$. By the *order* of a monomial $s_1^{\alpha_1} ... s_r^{\alpha_r}$ we mean the sum of the exponents $\alpha_1 + ... + \alpha_r$. By *multiplicity* $\mu(f(\mathbf{s}))$ *of* $f(\mathbf{s})$ *at* $\mathbf{0} = (0, \ldots, 0)$ we always mean the minimal order of non-zero monomials appearing in the Taylor expansion of $f(\mathbf{s})$ at $\mathbf{0}$.

Definition 6.9 Let $f(\mathbf{s})$ be a meromorphic at $\mathbf{0}$ function. Define the *multiplicity* $\mu(f(\mathbf{s}))$ of $f(\mathbf{s})$ at $\mathbf{0}$ as

$$\mu(f(\mathbf{s})) = \mu(g_1(\mathbf{s})) - \mu(g_2(\mathbf{s}))$$

where $g_1(\mathbf{s})$ and $g_2(\mathbf{s})$ are two coprime elements in $\mathbf{C}[[\mathbf{s}]]$ such that $f = g_1/g_2$.

Remark 6.10 It is easy to show that for any two meromorphic at $\mathbf{0}$ functions $f_1(\mathbf{s})$ and $f_2(\mathbf{s})$, one has

(i) $\mu(f_1 \cdot f_2) = \mu(f_1) + \mu(f_2)$ (in particular, one can omit "coprime" in Definition 6.9);

(ii) $\mu(f_1 + f_2) \geq \min\{\mu(f_1), \mu(f_2)\}$;

(iii) $\mu(f_1 + f_2) = \mu(f_1)$ if $\mu(f_2) > \mu(f_1)$.

Using the properties 6.10(i)-(ii), one immediately obtains from Definition 6.8 the following:

Proposition 6.11 *Let $f_1(\mathbf{s})$ and $f_2(\mathbf{s})$ be two analytic at $\mathbf{0}$ functions, $l(\mathbf{s})$ a homogeneous linear function, $\gamma = (\gamma_1, \dots, \gamma_r) \in \mathbf{C}^r$ an arbitrary complex vector with $l(\gamma) \neq 0$, and $g(\mathbf{s}) := f_1(\mathbf{s})/f_2(\mathbf{s})$. Then the multiplicity of the function*

$$\tilde{g}(\mathbf{s}) := \left(\frac{\partial}{\partial z}\right)^k g(\mathbf{s} + z \cdot \gamma)|_{z=-l(\mathbf{s})/l(\gamma)}$$

at $\mathbf{0}$ is at least $\mu(g) - k$, if

$$f_2(\mathbf{s} + z \cdot \gamma)|_{z=-l(\mathbf{s})/l(\gamma)}$$

is not identically zero.

Let $\Gamma \subset \mathbf{Z}^r$ be a sublattice, $\Gamma_{\mathbf{R}} \subset \mathbf{R}^r$ (resp. $\Gamma_{\mathbf{C}} \subset \mathbf{C}^r$) the scalar extension of Γ to a $\mathbf{R}$-subspace (resp. to a $\mathbf{C}$-subspace). We always assume that $\Gamma_{\mathbf{R}} \cap \mathbf{R}^r_{\geq 0} = 0$. We set $V_{\mathbf{R}} := \mathbf{R}^r/\Gamma_{\mathbf{R}}$ and $V_{\mathbf{C}} := \mathbf{C}^r/\Gamma_{\mathbf{C}}$. Denote by ψ the canonical $\mathbf{C}$-linear projection $\mathbf{C}^r \to V_{\mathbf{C}}$.

Definition 6.12 A complex analytic function $f(\mathbf{s}) = f(s_1, \dots, s_r) : U \to \mathbf{C}$ defined on an open subset $U \subset \mathbf{C}^r$ is said to *descend to* $V_{\mathbf{C}}$ if for any vector $\alpha \in \Gamma_{\mathbf{C}}$ and any $\mathbf{u} = (u_1, \dots, u_r) \in U$ one has

$$f(\mathbf{u} + z \cdot \alpha) = f(\mathbf{u}) \text{ for all } \{z \in \mathbf{C} \,:\, \mathbf{u} + z \cdot \alpha \in U\}.$$

Remark 6.13 By definition, if $f(\mathbf{s})$ descends to $V_{\mathbf{C}}$, then there exists an analytic function g on $\psi(U) \subset V_{\mathbf{C}}$ such that $f = g \circ \psi$. Using Cauchy-Riemann equations, one immediately obtains that f descends to $V_{\mathbf{C}}$ if and only if for any vector $\alpha \in \Gamma_{\mathbf{R}}$ and any $\mathbf{u} = (u_1, \dots, u_r) \in U$, one has

$$f(\mathbf{u} + iy \cdot \alpha) = f(\mathbf{u}) \text{ for all } \{y \in \mathbf{R} \,:\, \mathbf{u} + iy \cdot \alpha \in U\}.$$

Definition 6.14 Let U be an open subset in $\mathbf{R}^r$. By a *tube domain* $U_{\mathbf{C}}$ *over* U we mean the set

$$U_{\mathbf{C}} := \{\mathbf{z} \in \mathbf{C}^r \ : \ \mathrm{Re}(\mathbf{z}) \in U\}.$$

The following statement can be found in [17] (Prop. 6 on p. 122):

Theorem 6.15 *Let $U \subset \mathbf{R}^r$ (with $r \geq 2$) be a connected open subset. Then any function $f(\mathbf{z})$ which is analytic in $U_{\mathbf{C}}$ extends to an analytic function in $\hat{U}_{\mathbf{C}}$, where $\hat{U}_{\mathbf{C}}$ is the tube domain over the convex hull $\hat{U}$ of U.*

Definition 6.16 An analytic function $W(\mathbf{s})$ in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>0}$ is called *good with respect to* Γ if it satisfies the following conditions:

(i) $W(\mathbf{s})$ descends to $V_{\mathbf{C}}$;

(ii) There exist pairwise coprime linear homogeneous polynomials

$$l_1(\mathbf{s}), \ldots, l_p(\mathbf{s}) \in \mathbf{R}[\mathbf{s}]$$

and positive integers $k_1, \ldots, k_p$ such that for every $j \in \{1, \ldots, p\}$ the linear form $l_j(\mathbf{s})$ descends to $V_{\mathbf{C}}$, $l_j(\mathbf{s})$ does not vanish for $\mathbf{s} \in \mathbf{R}^n_{>0}$, and

$$P(\mathbf{s}) = W(\mathbf{s}) \cdot \prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})$$

is analytic for $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>-\delta_0}$ for some $\delta_0 > 0$.

(iii) There exist a non-zero complex number $C(W)$ and a decomposition of $P(\mathbf{s})$ into the sum

$$P(\mathbf{s}) = P_0(\mathbf{s}) + P_1(\mathbf{s})$$

so that $P_0(\mathbf{s})$ is a homogeneous polynomial of degree $\mu(P)$, $P_1(\mathbf{s})$ is an analytic function in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>-\delta_0}$ with $\mu(P_1) > \mu(P_0)$, both functions P_0, P_1 descend to $V_{\mathbf{C}}$, and

$$\frac{P_0(\mathbf{s})}{\prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})} = C(W) \cdot \mathcal{X}_{\Lambda}(\psi(\mathbf{s})),$$

where $\mathcal{X}_{\Lambda}$ is the $\mathcal{X}$-function of the cone $\Lambda = \psi(\mathbf{R}^r_{\geq 0}) \subset V_{\mathbf{R}}$;

Definition 6.17 If $W(\mathbf{s})$ is a good with respect to Γ as above, then the meromorphic function

$$\frac{P_0(\mathbf{s})}{\prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})}$$

will be called the *principal part of* $W(\mathbf{s})$ *at* $\mathbf{0}$ and the non-zero constant $C(W)$ the *principal coefficient of* $W(\mathbf{s})$ *at* $\mathbf{0}$.

Suppose that $\operatorname{rk}\Gamma = t < r-1$. Let $\gamma \in \mathbf{Z}^r$ be an element which is not contained in Γ, $\tilde{\Gamma} := \Gamma \oplus \mathbf{Z} < \gamma >$, $\tilde{\Gamma}_{\mathbf{R}} := \Gamma_{\mathbf{R}} \oplus \mathbf{R} < \gamma >$, $\tilde{V}_{\mathbf{R}} := \mathbf{R}^r/\tilde{\Gamma}_{\mathbf{R}}$ and $\tilde{V}_{\mathbf{C}} := \mathbf{C}^r/\tilde{\Gamma}_{\mathbf{C}}$.

The following easy statement will be helpful in the sequel:

Proposition 6.18 *Let $f(\mathbf{s})$ be an analytic at $\mathbf{0}$ function, $l(\mathbf{s})$ a homogeneous linear function such that $l(\gamma) \neq 0$. Assume that $f(\mathbf{s})$ and $l(\mathbf{s})$ descend to $V_{\mathbf{C}}$. Then*

$$\tilde{f}(\mathbf{s}) := f\left(\mathbf{s} - \frac{l(\mathbf{s})}{l(\gamma)} \cdot \gamma\right)$$

descends to $\tilde{V}_{\mathbf{C}}$.

Theorem 6.19 *Let $W(\mathbf{s})$ be a good function with respect to Γ as above,*

$$\Phi(\mathbf{s}) = \prod_{j\,:\,l_j(\gamma)=0} l_j^{k_j}(\mathbf{s})$$

the product of those linear forms $l_j(\mathbf{s})$ $(j \in \{1, \ldots, p\})$ which vanish on γ. Assume that $\tilde{\Gamma}_{\mathbf{R}} \cap \mathbf{R}^r_{\geq 0} = 0$ and the following statements hold:

(i) *The integral*

$$\tilde{W}(\mathbf{s}) := \int_{\operatorname{Re}(z)=0} W(\mathbf{s} + z \cdot \gamma) dz, \quad z \in \mathbf{C}$$

converges absolutely and uniformly on any compact $\mathbf{K} \subset \mathbf{C}^r$ in the domain $\operatorname{Re}(\mathbf{s}) \in \mathbf{R}^r_{>0}$*;*

(ii) *There exists a $\delta > 0$ such that the function $W(\mathbf{s} + z \cdot \gamma)$ is defined and analytic in a tube domain $U_{\mathbf{C}}$ over an open neighborhood U of $\mathbf{0}$ for all z with* $\operatorname{Re}(z) = \delta$ *and the integral*

$$\int_{\operatorname{Re}(z)=\delta} \Phi(\mathbf{s}) \cdot W(\mathbf{s} + z \cdot \gamma) dz$$

converges absolutely and uniformly in any compact which is contained in the same tube domain $U_{\mathbf{C}}$. Moreover, the multiplicity of the meromorphic function

$$\tilde{W}_\delta(\mathbf{s}) := \int_{\mathrm{Re}(z)=\delta} W(\mathbf{s} + z\cdot\gamma)dz$$

at $\mathbf{0}$ *is at least* $1 + \mathrm{rk}\,\tilde{\Gamma} - r$*;*

(iii) *For* $\mathbf{s}$ *with* $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>0}$*, the function*

$$\phi(N, \mathbf{s}) = \sup_{0\le \mathrm{Re}(z)\le\delta,\ |\mathrm{Im}(z)|=N} |W(\mathbf{s} + z\cdot\gamma)|$$

tends to 0 *as* $N \to +\infty$*, uniformly in any compact* $\mathbf{K}$ *contained in the tube domain over* $\mathbf{R}^r_{>0}$*.*

Then $\tilde{W}(\mathbf{s})$ *is a good function with respect to* $\tilde{\Gamma}$ *and* $C(\tilde{W}) = 2\pi i\cdot C(W)$*.*

Proof. Assume that $l_j(\gamma) < 0$ for $j = 1, \ldots, p_1$, $l_j(\gamma) = 0$ for $j = p_1+1, \ldots, p_2$, and $l_j(\gamma) > 0$ for $j = p_2 + 1, \ldots, p$. In particular, one has

$$\Phi(\mathbf{s}) = \prod_{j=p_1+1}^{p_2} l_j^{k_j}(\mathbf{s}).$$

Denote by z_j the solution of the equation

$$l_j(\mathbf{s}) + zl_j(\gamma) = 0, \quad j = 1, \ldots, p_1.$$

Let U^+ be the intersection of $\mathbf{R}^r_{>0}$ with an open neighborhood U of $\mathbf{0}$, such that

$$\Phi(\mathbf{s})\cdot\tilde{W}_\delta(\mathbf{s})$$

is analytic in $U_{\mathbf{C}}$. Then both functions $\tilde{W}_\delta(\mathbf{s})$ and $\tilde{W}(\mathbf{s})$ are analytic in $U^+_{\mathbf{C}}$. Moreover, the integral formulas for $\tilde{W}_\delta(\mathbf{s})$ and $\tilde{W}(\mathbf{s})$ show that the equalities $\tilde{W}_\delta(\mathbf{u} + iy\cdot\gamma) = \tilde{W}_\delta(\mathbf{u})$ and $\tilde{W}(\mathbf{u} + iy\cdot\gamma) = \tilde{W}(\mathbf{u})$ hold for any $y \in \mathbf{R}$ and $\mathbf{u}, \mathbf{u} + iy\cdot\gamma \in U^+_{\mathbf{C}}$. Therefore, both functions $\tilde{W}_\delta(\mathbf{s})$ and $\tilde{W}(\mathbf{s})$ descend to $\tilde{V}_{\mathbf{C}}$ (see Remark 6.13).

For $N \gg 0$, we can apply the residue theorem and obtain

$$\int_{-N}^{N} W(\mathbf{s} + it\cdot\gamma)dt + \int_0^\delta W(\mathbf{s} + (iN + y)\gamma)dy -$$

$$-\int_{-N}^{N} W(\mathbf{s}+(it+\delta)\cdot\gamma)dt - \int_{0}^{\delta} W(\mathbf{s}+(-iN+y)\gamma)dy =$$

$$= -2\pi i \cdot \sum_{j=1}^{p_1} \mathrm{Res}_{z=z_j} W(\mathbf{s}+z\cdot\gamma)$$

for all $\mathbf{s} \in U_{\mathbf{C}}^{+}$. By (iii), we have

$$\lim_{N\to+\infty} \int_{0}^{\delta} W(\mathbf{s}+(iN+y)\gamma)dy = 0$$

and

$$\lim_{N\to+\infty} \int_{0}^{\delta} W(\mathbf{s}+(-iN+y)\gamma)dy = 0$$

uniformly for all $\mathbf{s}$ contained in any compact $\mathbf{K} \subset U_{\mathbf{C}}^{+}$. By (i) and (ii), we have

$$\tilde{W}(\mathbf{s}) = \lim_{N\to+\infty} \int_{-N}^{N} W(\mathbf{s}+it\cdot\gamma)dt$$

and

$$\tilde{W}_{\delta}(\mathbf{s}) = \lim_{N\to+\infty} \int_{-N}^{N} W(\mathbf{s}+(it+\delta)\cdot\gamma)dt$$

uniformly for all $\mathbf{s}$ contained in a compact $\mathbf{K} \subset U_{\mathbf{C}}^{+}$. Therefore,

$$\tilde{W}(\mathbf{s}) - \tilde{W}_{\delta}(\mathbf{s}) = -2\pi i \cdot \sum_{j=1}^{p_1} \mathrm{Res}_{z=z_j} W(\mathbf{s}+z\cdot\gamma)$$

for all $\mathbf{s} \in U_{\mathbf{C}}^{+}$.

We denote by $U(\gamma)_{\mathbf{C}}$ the open subset of $U_{\mathbf{C}}$ defined by the inequalities

$$\frac{l_j(\mathbf{s})}{l_j(\gamma)} \neq \frac{l_m(\mathbf{s})}{l_m(\gamma)} \text{ for all } j \neq m, \quad j,m \in \{1,\dots,p\}.$$

The open set $U(\gamma)_{\mathbf{C}}$ is non-empty, since we assume that $g.c.d.(l_j, l_m) = 1$ for $j \neq m$. For $\mathbf{s} \in U(\gamma)_{\mathbf{C}}$, we have

$$\mathrm{Res}_{z=z_j} W(\mathbf{s}+z\cdot\gamma) = \frac{1}{(k_j-1)!} \left(\frac{\partial}{\partial z}\right)^{k_j-1} \frac{l_j(\mathbf{s}+z\cdot\gamma)^{k_j} P(\mathbf{s}+z\cdot\gamma)}{l_j^{k_j}(\gamma)\cdot\prod_{m=1}^{p} l_m^{k_m}(\mathbf{s}+z\cdot\gamma)}|_{z=z_j},$$

where

$$z_j = -\frac{l_j(\mathbf{s})}{l_j(\gamma)}.$$

Let

$$W(\mathbf{s}) \cdot \prod_{j=1}^{p} l_j^{k_j}(\mathbf{s}) = P(\mathbf{s}) = P_0(\mathbf{s}) + P_1(\mathbf{s}),$$

where $P_0(\mathbf{s})$ is a uniquely determined homogeneous polynomial and $P_1(\mathbf{s})$ is an analytic in the tube domain over $\mathbf{R}^r_{>-\delta_0}$ function such that $\mu(P) = \mu(P_0) < \mu(P_1)$ and

$$\frac{P_0(\mathbf{s})}{\prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})} = C(W) \cdot \mathcal{X}_\Lambda(\mathbf{s})$$

($\mathcal{X}_\Lambda(\mathbf{s})$ is the $\mathcal{X}$-function of the cone $\Lambda = \psi(\mathbf{R}^r_{\geq 0})$). We set

$$R_0(\mathbf{s}) := \frac{P_0(\mathbf{s})}{\prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})}, \quad R_1(\mathbf{s}) := \frac{P_1(\mathbf{s})}{\prod_{j=1}^{p} l_j^{k_j}(\mathbf{s})}.$$

Then $\mu(W) = \mu(R_0) < \mu(R_1)$. Moreover, $\mu(W) = -\mathrm{dim} V_{\mathbf{R}} = r - \mathrm{rk}\,\Gamma$. Define

$$\tilde{R}_0(\mathbf{s}) := -2\pi i \cdot \sum_{j=1}^{p_1} \mathrm{Res}_{z=z_j} R_0(\mathbf{s} + z \cdot \gamma)$$

and

$$\tilde{R}_1(\mathbf{s}) := -2\pi i \cdot \sum_{j=1}^{p_1} \mathrm{Res}_{z=z_j} R_1(\mathbf{s} + z \cdot \gamma).$$

By Proposition 6.11, we have $\mu(\tilde{R}_1) \geq 1 + \mu(R_1) \geq 2 + \mu(R_0) = 1 + \mathrm{rk}\,\tilde{\Gamma} - r$. We claim that

$$\tilde{R}_0(\mathbf{s}) = 2\pi i \cdot C(W) \mathcal{X}_{\tilde{\Lambda}}(\tilde{\psi}(\mathbf{s})).$$

(in particular, $\mu(\tilde{R}_0) = \mu(R_0) + 1 = \mathrm{rk}\,\tilde{\Gamma} - r$). Indeed, repeating for $\mathcal{X}_\Lambda(\psi(\mathbf{s}))$ the same arguments as for $W(\mathbf{s})$, we obtain

$$\int_{\mathrm{Re}(z)=0} \mathcal{X}_\Lambda(\psi(\mathbf{s} + z \cdot \gamma)) dz - \int_{\mathrm{Re}(z)=\delta} \mathcal{X}_\Lambda(\psi(\mathbf{s} + z \cdot \gamma)) dz$$

$$= -2\pi i \cdot \sum_{j=1}^{k_1} \mathrm{Res}_{z=z_j} \mathcal{X}_\Lambda(\psi(\mathbf{s} + z_j \cdot \gamma)).$$

Moving the contour of integration $\mathrm{Re}(z) = \delta$ ($\delta \to +\infty$), by residue theorem, we obtain

$$\int_{\mathrm{Re}(z)=\delta} \mathcal{X}_\Lambda(\psi(\mathbf{s} + z \cdot \gamma)) dz = 0.$$

On the other hand,

$$\mathcal{X}_{\tilde{\Lambda}}(\tilde{\psi}(\mathbf{s})) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=0} \mathcal{X}_{\Lambda}(\psi(\mathbf{s}+z\cdot\gamma))dz$$

(see Theorem 5.6).

By Proposition 6.11, we have $\mu(\tilde{R}_1) > 1+\mu(R_1) \geq 2+\mu(R_0) = 1+\mathrm{rk}\,\tilde{\Gamma}-r$.

By 6.9(iii), using the decomposition

$$\tilde{W}(\mathbf{s}) = \tilde{W}_\delta(\mathbf{s}) + \tilde{R}_0(\mathbf{s}) + \tilde{R}_1(\mathbf{s})$$

and our assumption $\mu(W_\delta) \geq 1 + \mathrm{rk}\,\tilde{\Gamma} - r$ in (ii), we obtain that $\mu(\tilde{W}) = \mu(\tilde{R}_0) = \mathrm{rk}\,\tilde{\Gamma} - r$.

By 6.18, the linear forms

$$h_{m,j}(\mathbf{s}) := l_m(\mathbf{s}+z_j\cdot\gamma) = l_m(\mathbf{s}) - \frac{l_j(\mathbf{s})}{l_j(\gamma)}l_m(\gamma)$$

and the analytic in the domain $U(\gamma)_{\mathbf{C}}$ functions

$$\mathrm{Res}_{z=z_j}W(\mathbf{s}+z\cdot\gamma), \ \ \mathrm{Res}_{z=z_j}R_0(\mathbf{s}+z\cdot\gamma)$$

descend to $\tilde{V}_{\mathbf{C}}$. For any $j \in \{1,\ldots,p_1\}$, let us denote

$$Q_j(\mathbf{s}) = \prod_{m\neq j, m=1}^{p} h_{m,j}^{k_m}(\mathbf{s}).$$

It is clear that

$$Q_j^{k_j}(\mathbf{s})\cdot \mathrm{Res}_{z=z_j}W(\mathbf{s}+z\cdot\gamma) \ \text{ and } \ Q_j^{k_j}(\mathbf{s})\cdot \mathrm{Res}_{z=z_j}R_0(\mathbf{s}+z\cdot\gamma)$$

are analytic at $\mathbf{0}$ and the polynomial $\Phi(\mathbf{s})$ divides each $Q_j(\mathbf{s})$. So we obtain that

$$\tilde{W}(\mathbf{s})\prod_{j=1}^{p_1} Q_j^{k_j}(\mathbf{s}) = \left(\tilde{W}_\delta(\mathbf{s}) - 2\pi i\cdot\sum_{j=1}^{p_1}\mathrm{Res}_{z=z_j}W(\mathbf{s}+z\cdot\gamma)\right)\prod_{j=1}^{p_1} Q_j^{k_j}(\mathbf{s})$$

and

$$\tilde{R}_0(\mathbf{s})\prod_{j=1}^{p_1} Q_j^{k_j}(\mathbf{s}) = \left(-2\pi i\cdot\sum_{j=1}^{p_1}\mathrm{Res}_{z=z_j}R_0(\mathbf{s}+z\cdot\gamma)\right)\prod_{j=1}^{p_1} Q_j^{k_j}(\mathbf{s})$$

are analytic in $U_{\mathbf{C}}$.

Define the set $\{\tilde{l}_1(\mathbf{s}), \dots, \tilde{l}_q(\mathbf{s})\}$ as a subset of pairwise coprime elements in the set of homogeneous linear forms $\{h_{m,j}(\mathbf{s})\}$ ($m \in \{1, \dots, p\}$, $j \in \{1, \dots, p_1\}$) such that there exist positive integers $n_1, \dots, n_q$ and a representation of the meromorphic functions $\tilde{W}(\mathbf{s})$ and $\tilde{R}_0(\mathbf{s})$ as quotients

$$\tilde{W}(\mathbf{s}) = \frac{\tilde{P}(\mathbf{s})}{\prod_{j=1}^q \tilde{l}_j^{n_j}(\mathbf{s})}, \quad \tilde{R}_0(\mathbf{s}) = \frac{\tilde{P}_0(\mathbf{s})}{\prod_{j=1}^q \tilde{l}_j^{n_j}(\mathbf{s})},$$

where $\tilde{P}(\mathbf{s})$ is analytic at $\mathbf{0}$, $\tilde{P}_0(\mathbf{s})$ is a homogeneous polynomial and none of the forms $\tilde{l}_1(\mathbf{s}), \dots, \tilde{l}_q(\mathbf{s})$ vanishes for $\mathbf{s} \in \mathbf{R}^r_{>0}$ (the last property can be achieved, because both functions $\tilde{W}(\mathbf{s})$ and $\tilde{R}_0(\mathbf{s})$ are analytic in $U^+_{\mathbf{C}}$).

Define

$$\tilde{P}_1(\mathbf{s}) = \Big(\tilde{W}_\delta(\mathbf{s}) + \tilde{R}_1(\mathbf{s})\Big) \cdot \prod_{j=1}^q \tilde{l}_j^{n_j}(\mathbf{s}).$$

Then

$$\tilde{P}(\mathbf{s}) = \tilde{P}_0(\mathbf{s}) + \tilde{P}_1(\mathbf{s})$$

where $\tilde{P}_0(\mathbf{s})$ is a homogeneous polynomial and $\tilde{P}_1(\mathbf{s})$ is an analytic in $U_{\mathbf{C}}$ function such that $\mu(\tilde{P}) = \mu(\tilde{P}_0) < \mu(\tilde{P}_1)$ and

$$\frac{\tilde{P}_0(\mathbf{s})}{\prod_{j=1}^q \tilde{l}_j^{n_j}(\mathbf{s})} = 2\pi i \cdot C(W) \cdot \mathcal{X}_{\tilde{\Lambda}}(\tilde{\psi}(\mathbf{s})).$$

For sufficiently small positive δ_0, the domain $\mathbf{R}^r_{>-\delta_0}$ is contained in the convex hull of $U \cup \mathbf{R}^r_{>0}$. By 6.15,

$$\prod_{j=1}^q \tilde{l}_j^{n_j}(\mathbf{s})\tilde{W}(\mathbf{s})$$

is analytic in $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{-\delta_0}$. This proves that $\tilde{W}(\mathbf{s})$ is a good function with respect to $\tilde{\Gamma}$. □

7 Main theorem

Let us set

$$W_\Sigma(\mathbf{s}) := Z_\Sigma(\varphi_{\mathbf{s}} + \varphi_\Sigma) = Z_\Sigma(s_1 + 1, \dots, s_r + 1).$$

By Theorem 4.2, $W_\Sigma(\mathbf{s})$ is an analytic function in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>0}$.

Theorem 7.1 *The analytic function $W_\Sigma(\mathbf{s})$ is good with respect to the lattice $M^G \subset PL(\Sigma)^G = \mathbf{Z}^r$.*

Proof. By Theorem 4.4, we have the following integral representation for $Z_\Sigma(\mathbf{s})$ in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>1}$

$$Z_\Sigma(\mathbf{s}) = \frac{1}{(2\pi)^t b_S(T)} \int_{(T(\mathbf{A}_K)/T(K))^*} \hat{H}_\Sigma(\chi, -\mathbf{s}) d\chi$$

We need only to consider characters χ which are trivial on the maximal compact subgroup $\mathbf{K}_T \subset T^1(\mathbf{A}_K)$, because for all other characters the Fourier transform $\hat{H}_\Sigma(\chi, -\mathbf{s})$ vanishes. Choosing a non-canonical splitting of characters corresponding to some splitting of the sequence

$$0 \to T^1(\mathbf{A}_K) \to T(\mathbf{A}_K) \to T(\mathbf{A}_K)/T^1(\mathbf{A}_K) \to 0$$

we obtain

$$Z_\Sigma(\mathbf{s}) = \frac{1}{(2\pi)^t b_S(T)} \int_{(T(\mathbf{A}_K)/T^1(\mathbf{A}_K))^*} d\chi_y \int_{(T^1(\mathbf{A}_K)/T(K))^*} \hat{H}_\Sigma(\chi, -\mathbf{s}) d\chi_l$$

We have an isomorphism $M^G_{\mathbf{R}} \simeq (T(\mathbf{A}_K)/T^1(\mathbf{A}_K))^*$ and the measure $d\chi_y$ coincides with the usual Lebesgue measure on $M^G_{\mathbf{R}}$. Recall that a character $\chi \in (T(\mathbf{A}_K)/T(K))^*$ defines r Hecke characters $\chi_1, ..., \chi_r$ of the idele groups $\mathbf{G}_m(\mathbf{A}_{K_j})$. In particular, we get r characters $\chi_{1,y}, ..., \chi_{r,y}$. We have an embedding $M^G \subset PL(\Sigma)^G$, which together with explicit formulas for Fourier transforms of local heights shows that the integral

$$A_\Sigma(\mathbf{s}, \chi_y) := \frac{1}{b_S(T)} \int_{(T^1(\mathbf{A}_K)/T(K))^*} \hat{H}_\Sigma(\chi, -(\mathbf{s} + \mathbf{1})) d\chi_l$$

is a function on $PL(\Sigma)^G_{\mathbf{C}}$ and we have

$$A_\Sigma(\mathbf{s}, \chi_y) = A_\Sigma(\mathbf{s} + i\mathbf{y}) = A_\Sigma(s_1 + iy_1, ..., s_r + iy_r).$$

Denote by $\Gamma := M^G$ the lattice of K-rational characters of T. Let t be the rank of Γ. The case $t = 0$ corresponds to an anisotropic torus T. It has been considered in [2]. So we assume $t > 0$.

For any element $\gamma \in \Gamma \subset \mathbf{Z}^r$ we denote by $l(\gamma)$ the number of its coordinates which are zero ($0 \leq l(\gamma) \leq r$). Let $l(\Gamma)$ be the minimum of $l(\gamma)$

among $\gamma \in \Gamma$. Notice that $l(\Gamma) \leq r - t - 1$. Indeed, if we had $l(\Gamma) \geq r - t$, then M^G would be contained in the intersection of $r - t$ linear coordinate hyperplanes $s_j = 0$ (the latter contradicts the condition $M^G_{\mathbf{R}} \cap \mathbf{R}^r_{\geq 0} = 0$). We can always choose a $\mathbf{Z}$-basis $\gamma^1, \ldots, \gamma^t$ of Γ in such a way that $l(\Gamma) = l(\gamma^u)$ $(u = 1, \ldots, t)$. Without loss of generality we assume that Γ is contained in the intersection of coordinate hyperplanes $s_j = 0$, $j \in \{1, \ldots, l(\Gamma)\}$. We set

$$\Phi(\mathbf{s}) := \prod_{j=1}^{l(\Gamma)} s_j.$$

For any $u \leq t$ we define a subgroup $\Gamma^{(u)} \subset \Gamma$ of rank u as

$$\Gamma^{(u)} := \bigoplus_{j=1}^{u} \mathbf{Z} < \gamma^j > .$$

We introduce some auxiliary functions

$$W^{(u)}_\Sigma(\mathbf{s}) = \int_{\Gamma^{(u)}_{\mathbf{R}}} A_\Sigma(\mathbf{s} + i\mathbf{y}^{(u)}) \mathbf{dy}^{(u)}$$

where $\mathbf{dy}^{(u)}$ is the induced Lebesgue measure on $\Gamma^{(u)}_{\mathbf{R}} \subset \mathbf{R}^r$, normalized by $\Gamma^{(u)}$. Denote $V^{(u)}_{\mathbf{C}} = \mathbf{C}^r / \Gamma^{(u)}_{\mathbf{C}}$. We prove by induction that $W^{(u)}_\Sigma(\mathbf{s})$ is good with respect to $\Gamma^{(u)} \subset \mathbf{Z}^r$.

By 4.6, $W^{(u)}_\Sigma(\mathbf{s})$ is an analytic function in the domain $\mathrm{Re}(\mathbf{s}) \in \mathbf{R}^r_{>0}$.

There exist $\delta_1, ..., \delta_t > 0$ such that the integral

$$\int_{\mathrm{Re}(z) = \delta_u} \Phi(\mathbf{s}) \cdot W^{(u-1)}_\Sigma(\mathbf{s} + z \cdot \gamma^u) dz \ (u = 1, 2, \ldots, t)$$

converges absolutely and uniformly in any compact contained in a tube domain $U_{\mathbf{C}}$ over an open neighborhood U of $\mathbf{0}$. This can be seen as follows: For any ε with $0 < \varepsilon < 1/rd'$, where $d' = \dim M_{\mathbf{R},\infty}$, we can choose a ball $B_{e_1} \subset \mathbf{R}^r$ of radius e_1 around $\mathbf{0}$ such that for any ball $B_{e_2} \subset B_{e_1}$ of radius e_2 $(0 < e_2 < e_1)$ around 0 we can uniformly bound the Hecke L-functions $L_{K_j}(s_j + 1, \chi_j)$ appearing in $\hat{H}_\Sigma(\chi, \mathbf{s})$ by

$$c_j(e_2)(\|y(\chi_j)\| + |\mathrm{Im}(s_j)| + 1)^\varepsilon$$

with some constants $c_j(e_2)$ for all $\mathbf{s}$ in the domain $\mathrm{Re}(s_j) \in B_{e_1} \backslash B_{e_2}$ for $j = 1, ..., r$ (see 6.2). By 4.6, this assures the absolute and uniform convergence of the integral

$$\int_{\Gamma^{(u)}_{\mathbf{R}}} A_\Sigma(\mathbf{s} + i\mathbf{y}^{(u)})\mathbf{dy}^{(u)}$$

for all $\mathbf{s}$ contained in a compact in $\mathbf{C}^r$ such that $\mathrm{Re}(s_j) \in B_{\varepsilon_1} \backslash B_{\varepsilon_2}$ for $j = 1, ...r$. We know that the coordinates γ^u_j of the vectors $\gamma^u = (\gamma^u_1, ..., \gamma^u_r) \in \mathbf{Z}^r$ are not equal to zero for $l(\Gamma) < j \leq r$. Therefore, we can now choose such real $\delta_u > 0$ that $\delta_u \gamma^u_j$ are all contained in the *open* ball B_{e_1}. So there must exist some $e_2 > 0$ such that $\delta_u \gamma^u_j \notin B_{e_2}$ for all $u = 1, ..., t$ and all $l(\Gamma) < j \leq r$. It follows that there exists an open neighborhood of $\mathbf{0}$, such that for all $\mathbf{s}$ contained in this neighborhood we have $\mathrm{Re}(s_j + \delta_u \gamma^u_j) \in B_{e_1} \backslash B_{e_2}$ for all $l(\Gamma) < j \leq r$. Since we remove the remaining poles by multiplying with $\Phi(\mathbf{s})$ we obtain the absolute and uniform convergence of $W^{(u)}_\Sigma(\mathbf{s})$ to a holomorphic function in $\mathbf{s}$ in this neighborhood.

Moreover, the multiplicity of the meromorphic function

$$\tilde{W}^{(u)}_{\delta_u}(\mathbf{s}) := \int_{\mathrm{Re}(z)=\delta_u} W^{(u-1)}_\Sigma(\mathbf{s} + z \cdot \gamma_u) dz$$

at $\mathbf{0}$ is at least

$$\mu(\Phi(\mathbf{s})) = 1 + t - r = 1 + \mathrm{rk}\,\Gamma - r \geq 1 + \mathrm{rk}\,\Gamma^{(u)} - r.$$

We apply Theorem 6.19. It is clear that

$$\sum_{\chi \in (T^1(\mathbf{A}_K/T(K)\mathbf{K}_T)^*} \hat{H}_\Sigma(\chi, -(s_1 - 1), \ldots, -(s_r - 1))$$

is good with respect to the trivial lattice $\Gamma = 0$ (The property (iii) follows from estimates 6.3 and 4.6). This concludes the proof.

□

Theorem 7.2 *Denote by* $\hat{H}_{\Sigma,S}(\chi, -\mathbf{s})$ *the multiplicative Fourier transform of the height function with respect to the measure* $\omega_{\Omega,S}$ *(see 2.2). The principal coefficient* $C(\Sigma)$ *of*

$$A_\Sigma(\mathbf{s}) = \frac{1}{b_S(T)} \int_{(T^1(\mathbf{A}_K/T(K)\mathbf{K}_T)^*} \hat{H}_{\Sigma,S}(\chi_l, -\mathbf{s}) d\chi_l$$

at $s_1 = ... = s_r = 1$ *is equal to* $\beta(\mathbf{P}_\Sigma)\tau_{\mathcal{K}}(\mathbf{P}_\Sigma)$.

Proof. We follow closely the exposition of the proof of theorem 3.4.6 in [2]. Since $M^G \hookrightarrow PL(\Sigma)^G$ we have an embedding of characters

$$(T(\mathbf{A}_K)/T^1(\mathbf{A}_K))^* = M^G_{\mathbf{R}} \hookrightarrow \prod_{j=1}^{r} (\mathbf{G}_m(\mathbf{A}_{K_j})/\mathbf{G}^1_m(\mathbf{A}_{K_j}))^*.$$

Recall that the kernel of

$$a^* : (T(\mathbf{A}_K)/T(\mathbf{A}_K))^* \to \prod_{j=1}^{r} (\mathbf{G}_m(\mathbf{A}_{K_j})/\mathbf{G}_m(\mathbf{A}_{K_j}))^*$$

is dual to the obstruction group to weak approximation $A(T) = T(\mathbf{A}_K)/\overline{T(K)}$. We have a splitting

$$\overline{T(K)} = \overline{T(K)}_S \times T(A_{K,S}).$$

Here we denoted by $\overline{T(K)}_S$ the image of $\overline{T(K)}$ in $\prod_{v\in S} T(K_v)$ and $T(A_{K,S}) = T(\mathbf{A}_K) \cap \prod_{v\notin S} T(K_v)$. The pole of the highest order r of $\hat{H}_{\Sigma,S}(\chi_l, -\mathbf{s})$ at $s_1 = ... = s_r = 1$ appears from characters χ_l such that the corresponding $\chi_1, ..., \chi_r$ are trivial characters of the groups $\mathbf{G}_m(\mathbf{A}_{K_j})/\mathbf{G}_m(K_j)$, i.e., χ_l is a character of the finite group $A(T) = \prod_{v\in S} T(K_v)/\overline{T(K)}_S$, and is trivial on the group $T(\mathbf{A}_{K,S})$.

For $\mathbf{s} \in \mathbf{R}^r_{>1}$ we can again apply the Poisson formula to $A(T)$. By 1.18, the order of $A(T)$ equals $\beta(\mathbf{P}_\Sigma)/i(T)$. We obtain

$$\frac{1}{b_S(T)} \sum_{\chi\in(A(T))^*} \hat{H}_{\Sigma,S}(\chi_l, -\mathbf{s}) = \frac{\beta(\mathbf{P}_\Sigma)}{i(T)b_S(T)} \int_{\overline{T(K)}} H_\Sigma(x, -\mathbf{s})\omega_{\Omega,S}$$

(see 1.18). We restrict to the line $s_1 = ... = s_r = s$ and we want to compute the limit

$$\lim_{s\to 1}(s-1)^r \int_{\overline{T(K)}} H_\Sigma(x, -\mathbf{s})\omega_{\Omega,S}.$$

We have

$$\int_{\overline{T(K)}} H_\Sigma(x, -\mathbf{s})\omega_{\Omega,S} = \tag{1}$$

$$= \int_{\overline{T(K)}_S} \prod_{v\in S} H_{\Sigma,v}(x_v, -\mathbf{s})\omega_{\Omega,v} \cdot \prod_{v\notin S} \int_{T(K_v)} H_{\Sigma,v}(x_v, -\mathbf{s}) d\mu_v$$

(recall that $\omega_{\Omega,v} = \prod_{v\in \mathrm{Val}(K)} d\mu_v$ and $d\mu_v = L_v(1, T; E/K)\omega_{\Omega,v}$ for all v and $L_v(1, T; E/K) = 1$ for $v \in S$).

From our calculations of the Fourier transform of local height functions for $v \notin S$ (3.10), we have

$$\prod_{v \notin S} \int_{T(K_v)} H_{\Sigma,v}(x_v, -\mathbf{s}) d\mu_v = \tag{2}$$

$$= L_S(s, T; E/K) \cdot L_S(s, T_{NS}; E/K) \prod_{v \notin S} Q_\Sigma(q_v^{-s}, \ldots, q_v^{-s}).$$

By 3.7,

$$\prod_{v \notin S} Q_\Sigma(q_v^{-s}, \ldots, q_v^{-s})$$

is an absolutely convergent Euler product for $s = 1$. Moreover, the limits

$$\lim_{s \to 1}(s-1)^t L_S(s, T; E/K)$$

$$\lim_{s \to 1}(s-1)^{(r-t)} L_S(s, T_{NS}; E/K)$$

exist and equal the non-zero constants $l_S(T)$ and $l_S^{-1}(\mathbf{P}_\Sigma)$ $(r = t + k)$. By 3.11,

$$\int_{\overline{T(K)}_S} \prod_{v \in S} H_{\Sigma,v}(x_v, -\mathbf{s}) \omega_{\Omega,v}$$

is absolutely convergent for $s_1 = ... = s_r = 1$. Using (1) and (2), we obtain:

$$\lim_{s \to 1}(s-1)^r \int_{\overline{T(K)}} H_\Sigma(x, -\mathbf{s}) \omega_{\Omega,S} = \tag{3}$$

$$= \frac{l_S(T)}{l_S(\mathbf{P}_\Sigma)} \int_{\overline{T(K)}_S} \prod_{v \in S} H_{\Sigma,v}(x_v, -\mathbf{s}) \omega_{\Omega,v} \cdot \prod_{v \notin S} Q_\Sigma(q_v^{-1}, \ldots, q_v^{-1}).$$

Now recall (3.10), that for $v \notin S$ we have

$$Q_\Sigma(q_v^{-1}, \ldots, q_v^{-1}) = \int_{T(K_v)} L_v^{-1}(1, T_{NS}; E/K) H_{\Sigma,v}(x_v, -\mathbf{1}) \omega_{\Omega,v}.$$

It was proved in [2] Proposition 3.4.4 that the restriction of the v-adic measure $\omega_{\mathcal{K},v}$ to $T(K_v) \subset \mathbf{P}_\Sigma(K_v)$ coincides with the measure

$$H_{\Sigma,v}(x, -\mathbf{1}) \omega_{\Omega,v}.$$

Here $\mathcal{K}$ is the canonical sheaf on the toric variety $\mathbf{P}_\Sigma$ metrized as above.

We also have

$$\int_{\overline{T(K)}_S} \prod_{v\in S} H_{\Sigma,v}(x_v, -\mathbf{1})\omega_{\Omega,v} = \int_{\overline{T(K)}_S} \prod_{v\in S} \omega_{\mathcal{K},v}. \tag{4}$$

Using the splitting $\overline{T(K)} = \overline{T(K)}_S \times T(A_{K,S})$ and multiplying the above equations we get

$$\int_{\overline{T(K)}} \omega_{\mathcal{K},S} = \int_{\overline{T(K)}_S} \prod_{v\in S} \omega_{\mathcal{K},v} \cdot \prod_{v\notin S} \int_{T(K_v)} L_v^{-1}(1, T_{NS}; E/K)\omega_{\mathcal{K},v}.$$

On the other hand, it was proved in [2], Proposition 3.4.5 that we have

$$\int_{\overline{T(K)}} \omega_{\mathcal{K},S} = \int_{\overline{\mathbf{P}_\Sigma(K)}} \omega_{\mathcal{K},S} = b_S(\mathbf{P}_\Sigma).$$

Therefore,

$$b_S(\mathbf{P}_\Sigma) = \int_{\overline{T(K)}_S} \prod_{v\in S} H_\Sigma(x, -\varphi_\Sigma)\omega_{\Omega,v} \cdot \prod_{v\notin S} Q_\Sigma(q_v^{-1}, \ldots, q_v^{-1}).$$

Collecting terms, we obtain

$$C(\Sigma) = \frac{\beta(\mathbf{P}_\Sigma)}{i(T)b_S(T)} \cdot \frac{l_S(T)}{l_S(\mathbf{P}_\Sigma)} \cdot b_S(\mathbf{P}_\Sigma).$$

By 2.4 and 2.9, we have the following equality

$$i(T)b_S(T) = h(T)l_S(T).$$

It remains to notice that we have an exact sequence of lattices

$$0 \to M^G \to PL(\Sigma)^G \to \mathrm{Pic}(\mathbf{P}_\Sigma) \to H^1(G, M) \to 0$$

and that the number $h(T) = |H^1(G, M)|$ appears in the integral formula for the $\mathcal{X}$-function of the cone $\Lambda_{\text{eff}} \subset \mathrm{Pic}(\mathbf{P}_\Sigma)$. We apply Theorem 5.5 and obtain that

$$W_\Sigma(\mathbf{s}) = \frac{1}{(2\pi)^t b_S(T)} \int_{M^G_{\mathbf{R}}} A_\Sigma(\mathbf{s} + i\mathbf{y})\mathbf{dy}$$

is good with respect to the lattice M^G and that

$$C(\Sigma) = \beta(\mathbf{P}_\Sigma)\tau_{\mathcal{K}}(\mathbf{P}_\Sigma)$$

is the principal coefficient of $W_\Sigma(\mathbf{s})$ at $\mathbf{0}$. □

Theorem 7.3 *There exists a $\delta > 0$ such that the height zeta-function $\zeta_\Sigma(s)$ obtained by restiction of the zeta-function $Z_\Sigma(\mathbf{s})$ to the complex line $s_j = \varphi(e_j) = s$ for all $j = 1, ..., r$ has a representation of the form*

$$\zeta_\Sigma(s) = \frac{\Theta(\Sigma)}{(s-1)^k} + \frac{g(s)}{(s-1)^{k-1}}$$

with $k = r - t = \mathrm{rk}\,\mathrm{Pic}(\mathbf{P}_\Sigma)$ and some holomorphic function $g(s)$ in the domain $\mathrm{Re}(s) > 1 - \delta$. Moreover, we have

$$\Theta(\Sigma) = \alpha(\mathbf{P}_\Sigma)\beta(\mathbf{P}_\Sigma)\tau_\mathcal{K}(\mathbf{P}_\Sigma).$$

Proof. Since $W_\Sigma(\mathbf{s})$ is good with respect to the lattice $M^G \subset \mathbf{Z}^r$, we have the following representation of $W_\Sigma(\mathbf{s})$ in a small open neighborhood of $\mathbf{0}$:

$$W_\Sigma(\mathbf{s}) = \frac{P(\mathbf{s})}{\prod_{j=1}^p l_j^{k_j}(\mathbf{s})}$$

where $P(\mathbf{s}) = P_0(\mathbf{s}) + P_1(\mathbf{s})$, $\mu(P_1) > \mu(P_0)$ and

$$\frac{P_0(\mathbf{s})}{\prod_{j=1}^p l_j^{k_j}(\mathbf{s})} = \beta(\mathbf{P}_\Sigma)\tau_\mathcal{K}(\mathbf{P}_\Sigma) \cdot \mathcal{X}_{\Lambda_{\mathrm{eff}}}(\psi(\mathbf{s})),$$

where $\mathcal{X}_{\Lambda_{\mathrm{eff}}}$ is the $\mathcal{X}$-function of the cone $\Lambda_{\mathrm{eff}} = \psi(\mathbf{R}^r_{\geq 0}) \subset \mathrm{Pic}(\mathbf{P}_\Sigma)_\mathbf{R}$.

If we restrict

$$\frac{P_0(\mathbf{s})}{\prod_{j=1}^p l_j^{k_j}(\mathbf{s})}$$

to the line $s_j = s - 1$ $(j = 1, \ldots, r)$, then we get the meromorphic function $\Theta(s-1)^{-k}$ with $\Theta = \alpha(\mathbf{P}_\Sigma)\beta(\mathbf{P}_\Sigma)\tau_\mathcal{K}(\mathbf{P}_\Sigma)$. Moreover, the order of the pole at $s = 1$ of the restriction of

$$\frac{P_1(\mathbf{s})}{\prod_{j=1}^p l_j^{k_j}(\mathbf{s})}$$

to the line $s_j = s - 1$ $(j = 1, \ldots, r)$ is less than k. Therefore, this restriction can be written as $g(s)/(s-1)^{k-1}$ for some analytic at $s = 1$ function $g(s)$.

Corollary 7.4 Let T be an algebraic torus and $\mathbf{P}_\Sigma$ its smooth projective compactification. Let k be the rank of $\mathrm{Pic}(\mathbf{P}_\Sigma)$. Then the number of K-rational points $x \in T(K)$ having the anticanonical height $H_{\mathcal{K}^{-1}}(x) \leq B$ has the asymptotic

$$N(T, \mathcal{K}^{-1}, B) = \frac{\Theta(\Sigma)}{(k-1)!} \cdot B(\log B)^{k-1}(1 + o(1)), \quad B \to \infty.$$

Proof. We apply a Tauberian theorem to $\zeta_\Sigma(s)$. □

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