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RATIONAL POINTS ON TORIC VARIETIES

V.V.BATYREV AND YU.TSCHINKEL

Abstract. We study the distribution of rational points on smooth compactifications of algebraic tori.

1. Introduction

Let X be a smooth Fano variety defined over a number field F and $-K_X$ its anticanonical line bundle. Over some finite extension one expects infinitely many rational points on X. One can introduce appropriate counting functions - height functions and study the asymptotic for the number of rational points of bounded height with growing height. In [1] Batyrev and Manin formulated conjectures describing such asymptotics. Let $\mathcal{L} = (L, \|\cdot\|_v)$ be a metrized very ample line bundle on X. Denote by $H_{\mathcal{L}} : X(F) \rightarrow \mathbb{R}_{\geq 0}$ the corresponding height function. Define the **height zeta function** associated with $\mathcal{L}$ as

$$Z_{\mathcal{L}}(s) = \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

Analytic properties of $Z_{\mathcal{L}}(s)$ determine the asymptotic behaviour of the number of F-rational points on X of bounded $\mathcal{L}$-height. Denote this number by $N(X, \mathcal{L}, B)$. It is an experimental fact that in order to get a good description of the asymptotics of $N(X, \mathcal{L}, B)$ for $B \rightarrow \infty$ in geometrical terms one should delete some Zariski closed subsets. Already for del Pezzo surfaces one has to deal with *accumulating subvarieties* - exceptional curves C are lines in the anticanonical embedding with an asymptotic $N(C, -\mathcal{K}_X|_C, B) \sim B^2$ while the conjecture predicts

$$N(U, -\mathcal{K}_X, B) \sim B(\log B)^{rk\ Pic(X)-1}$$

for some Zariski open U. One hopes that U can be chosen to be the complement of exceptional curves. In [5] geometric techniques were developed to prove upper bounds for the number of points on surfaces outside exceptional curves. One of the goals of the present paper is

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[2]Currently Junior Fellow of the Harvard Society of Fellows
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to show that it is still possible to get precise results if the accumulating subvarieties are contained in the divisor at infinity in a smooth compactification of an algebraic torus.
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2. Geometry

Let us first recall some terminology from the theory of toric varieties over algebraically closed fields [3, 6].

Definition 2.0.1. *A linear algebraic group T_F defined over a number field F is called a d-dimensional torus if its base extension $T = T_F \times_{Spec(F)} (Spec(\bar{F}))$ is isomorphic to $\mathbb{G}_m^d(\bar{F})$.*

The group of F-rational points of T_F is denoted by $T(F)$. We call T_F *split* if it is isomorphic to $\mathbb{G}_m^d(F)$ already over F. We call T_F *anisotropic* if it has no nontrivial F-split F-subtori. Examples of anisotropic tori are provided by norm equations. Equivariant compactifications of split tori are described by the following objects:

1. A finitely generated free $\mathbb{Z}$-module M of rank d and the dual module $N = \mathrm{Hom}(M, \mathbb{Z})$.
2. A complete d-dimensional fan Σ.

Definition 2.0.2. *A finite set Σ consisting of convex rational polyhedral cones in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ is called a* complete d-dimensional fan *if the following conditions are satisfied:*

(i) every cone $\sigma \in \Sigma$ contains $0 \in N_{\mathbb{R}}$;
(ii) every face σ' of a cone $\sigma \in \Sigma$ belongs to Σ;
(iii) the intersection of any two cones in Σ is a face of both cones;
(iv) $N_{\mathbf{R}}$ is the union of cones from Σ.

We denote by $\Sigma(i)$ the set of all i-dimensional cones in Σ. The toric variety X_Σ is obtained by glueing affine schemes

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$$

where $U_\sigma = Spec(F[M \cap \hat{\sigma}])$ and $\hat{\sigma}$ is the dual to σ cone. It contains T_F as a Zariski open subset. The toric variety X_Σ is smooth iff every cone in Σ is generated by a part of a $\mathbb{Z}$-basis of N.
There is a parallel theory of compactifications of nonsplit tori. In this case a T and Galois equivariant compactification of T_F is described by: M, N, Σ and an action of the Galois group on these objects.

Definition 2.0.3. A continuous function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ is called Σ-*piecewise linear* if the restriction of φ to every cone $\sigma \in \Sigma$ is a linear function. It is called *integral* if $\varphi(N) \subset \mathbb{Z}$. *We denote by $m_{\sigma,\varphi}$ the restriction of φ to σ considered as an element in M. The group of integral piecewise linear functions is denoted by $PL(\Sigma)$.*

Proposition 2.0.4. *The Picard group $\mathrm{Pic}(\mathbf{X}_\Sigma)$ is isomorphic to $PL(\Sigma)/M$ where elements of M are considered as globally linear integral functions on $N_{\mathbb{R}}$.*

Denote by $e_1, \dots, e_n$ the primitive integral generators of all 1-dimensional cones in Σ. They define T-invariant Weil divisors $D_1, \dots, D_n$ on X_Σ.

Proposition 2.0.5. *The piecewise linear function $\varphi \in PL(\Sigma)$ such that $\varphi(e_i) = 1$ for all i (if it exists) corresponds to the anticanonical class $-K_\Sigma$ in $Pic(X_\Sigma)$.*

Proposition 2.0.6. *The cone of effective divisors $N^1_{eff}(X_\Sigma)$ is generated by the classes $[D_1], \dots, [D_n]$.*

3. Heights

In this section we introduce metrizations on line bundles on toric varieties. The idea is that instead of studying $Z_{\mathcal{L}}(s)$ for some fixed metrized line bundle, one should consider the height zeta function as a function on the whole complexified Picard group. Therefore we need canonical simultaneous metrization on all line bundles. We don't know how to achieve this on arbitrary Fano varieties. We know how to do it in two examples: for flag varieties [4] and for toric varieties.

Let F be a global field and $\mathbb{A}_F$ the adele ring of F. Denote by $|\cdot|_v : F_v \to \mathbb{R}$ the standard norms on v-adic completions of F satisfying the product formula $\prod_v |a|_v = 1$ for all $a \in F^*$. Let $X = G/P$ be generalized flag variety. Here G is a semisimple linear algebraic group defined over F and P a parabolic subgroup containing a Borel subgroup B. Denote by $\pi : G \to G/P$ the natural projection. Denote by $X^*(P)$ the characters of P defined over F. There is an embedding of finite index

$$\begin{array}{ccc} X^*(P) & \to & Pic(G/P) \\ \chi & \mapsto & L_\chi \end{array}$$

where the line bundle L_χ is defined by

$$\Gamma(U, L_\chi) = \{f \in \Gamma(\pi^{-1}(U), \mathcal{O}_G) \mid f(pg) = \chi(p)f(g) \quad \forall p \in P \quad \forall g \in G\}.$$

Fix a compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v \subset G(\mathbb{A}_F)$ such that $G(\mathbb{A}_F) = B(\mathbb{A}_F)\mathbf{K}$. Let $k = k(x) \in \mathbf{K}$ be given by $\pi(k) = x \in G/P(F)$. For a section $s \in \Gamma(L_\chi)$ nonvanishing in x put $\|s(x)\|_v = |f(k)|_v$ where f is a function on G nonvanishing in x and satisfying $f(pg) = \chi(p)f(g)$ as above. The **height function** is defined as

$$H_{\mathcal{L}_\chi}(x) = \prod_v \|s(x)\|_v^{-1}.$$

The corresponding height zeta function is an Eisenstein series and its analytical properties can be summarized as follows: it is holomorphic in the domain $\rho_P + \mathfrak{a}_{\mathfrak{P}}^+$ where ρ_P is half of the sum of roots counted with their multiplicities and $\mathfrak{a}_P^+$ is the positive Weyl chamber. In geometric language it translates into holomorphy of the height zeta function inside the domain $-K_X + N^1_{eff}(X)$ for $X = G/P$ with simple poles at faces of the polyhedral cone $-K_X + \partial N^1_{eff}(X)$. The multiplicity of the pole at the vertex $-K_X$ equals $rk\ Pic(X)$. There are no accumulating subvarieties.

Let $X = X_\Sigma$ be a complete split toric variety obtained by glueing affine schemes $U_\sigma = Spec(F[M \cap \check{\sigma}])$. One can define a canonical covering of $X_\Sigma(F_v)$ by compact subsets $K_{\sigma,v} \subset U_\sigma(F_v)$ as follows:

$$K_{\sigma,v} = \{x_v \in U_\sigma(F_v) \mid |m(x_v)|_v \leq 1 \ \ for \ \ all \ \ m \in M \cap \check{\sigma}\}.$$

Here we have used the canonical embedding $T(F_v)/T(\mathcal{O}_v) \hookrightarrow N$ (resp. $N \otimes \mathbb{R}$ for archimedian v). For $\sigma, \tau \in \Sigma$ one has $K_{\sigma,v} \cap K_{\tau,v}$. Let L_φ be a line bundle on X_Σ given by a piecewise linear function $\varphi = \{m_{\sigma,\varphi}\} \in PL(\Sigma)$ and $s \in \Gamma(L_\varphi)$ a local section of L_φ nonvanishing in $x \in T(F) \subset X_\Sigma(F)$. Define the v-norm of s at $x_v \in K_{\sigma,v}$ by

$$\|s(x_v)\|_v = |s(x_v)/m_{\sigma,\varphi}(x_v)|_v.$$

This gives a $T(\mathcal{O}_v)$-invariant v-adic metric on $L_\varphi \otimes F_v$. The **height function** is defined as

$$H_{\mathcal{L}_\varphi}(x) = \prod_v \|s(x_v)\|_v^{-1}.$$

Example 3.1. *On $X_\Sigma = \mathbb{P}^{\Bbbk}$ the group of piecewise linear functions $PL(\Sigma)$ is 2- dimensional, $\varphi_s = (s_1, s_2)$ given by $\varphi(e_1) = s_1, \varphi(e_2) = s_2$. The corresponding local height function is*

$$H_v(x, s) = \{ \begin{array}{lll} |x|_v^{s_1} & if & |x|_v \geq 1 \\ |x|_v^{-s_2} & if & |x|_v \leq 1 \end{array}$$

Here we have $x \in T(F) = \mathbb{G}_{\gtrdot}(\mathbb{F}) \hookrightarrow \mathbb{F}$ and $|\cdot|_v$ is the standard valuation.

Let us point out that the introduced metrizations on line bundles for flag and for toric varieties are quite different. For $\mathbb{P}^{\Bbbk}$ considered as a flag variety the associated height zeta function is

$$Z(s) = \sum_{(n,m)=1} \frac{1}{(m^2+n^2)^{s/2}}$$

For $\mathbb{P}^{\Bbbk}$ considered as a compactification of $\mathbb{G}_{\gtrdot}$ the height zeta function is

$$Z_\Sigma(s) = 4\sum_n \frac{\phi(n)}{n^s} - 2$$

where $\phi(m)$ is the Euler function.
We want to extend the height function to the complexified Picard group $Pic(X_\Sigma)\otimes\mathbb{C}$. For a piecewise linear function $\varphi_s \in PL(\Sigma)\otimes\mathbb{C}$ given by $\varphi(e_i) = s_i$ for $s = (s_1, \dots, s_n)$ define the **complex local height function** as

$$H_{\Sigma,v}(x,s) = e^{\varphi_s(\bar{x}_v)\log q_v}.$$

Here q_v equals the cardinality of the residue field of F_v for non-archimedian v and $\log q_v = 1$ for archimedian valuations and $\bar{x}_v$ is the image of x_v in $N_v = N$. The **global complex height function** defined as a product

$$H_\Sigma(x,s) = \prod_v H_{\Sigma,v}(x,s)$$

extends naturally to $T_\Sigma(\mathbb{A}_F)$.

Proposition 3.0.7. *For $s \in \mathbb{Z}^{\ltimes}$ and $x \in X_\Sigma(F)$ the function $\prod_v H_{\Sigma,v}(x,s)$ coincides with the (classical) height function $H_{\mathcal{L}_{\varphi_s}}(x)$ defined through the canonical compact covering.*

Summing up: we have defined a function on the Zariski open subset $T \subset X_\Sigma$

$$H_\Sigma(x,s) : T(\mathbb{A}_F)\times Pic(X_\Sigma)_{\mathbb{C}} \to \mathbb{C}$$

invariant under the compact subgroup $\mathbf{K} \subset T(\mathbb{A}_F)$. The corresponding height zeta function

$$Z_\Sigma(t,s) = \sum_{x\in T(F)} H_\Sigma(tx, -s)$$

is invariant under $T(F)$. It is not difficult to see that the series converges absolutely and uniformly in the domain $Re(s) > 1 = \{Re(s_j) > 1 \ \ for\ j = 1, \dots, n\}$. The next step is to use harmonic analysis on the homogeneous space $T(\mathbb{A}_F)/T(F)$ and to try to establish the analytic properties of $Z_\Sigma(t,s)$. The accumulating subvarieties are contained in the complement $X_\Sigma - T$. All geometrical information is now encoded in the height H_Σ. Here is a precise version of Manin's conjecture in the case of toric varieties.

Conjecture 3.0.8. *The height zeta function $Z_\Sigma(t,s)$ is a function on $\frac{PL(\Sigma)}{M} \otimes \mathbb{C}$, it is holomorphic in the domain $Re(s) \in -K_\Sigma + N^1_{eff}(X_\Sigma)$ and has simple poles on codimension 1 faces of this cone.*

In general, we cannot hope to prove a functional equation and analytic continuation to the whole plane.

4. Fourier Analysis

4.1. **Adelization.** Denote by $T(\mathbb{A}_F)$ the adele group of T_F, by $T^1(\mathbb{A}_F)$ the subgroup of norm 1 adeles, by $\mathbf{KT(F)}$ the smallest closed subgroup of $T^1(\mathbb{A}_F)$ containing $T(F)$ and $K = \prod_v T(\mathcal{O}_v)$.
Let us formulate analogs of well known properties of ideles.

1. $T(\mathbb{A}_F)/T^1(\mathbb{A}_F) \simeq \mathbb{R}^{\smallsetminus}$ where r equals the rank of the group of F-rational characters of T_F,
2. $T^1(\mathbb{A}_F)/\mathbf{KT(F)}$ is isomorphic to a product of a finite abelian group (analog of the idele class group) and a connected compact abelian group,
3. $\prod_v T(\mathcal{O}_v) \cap T(F)$ is a finite group of torsion elements in $T(F)$ (this is the analog of the group of roots of unity).

4.2. **Poisson formula.** For a locally compact abelian group G denote by G^* its dual group of characters. For a function $f \in L^1(G)$ define the Fourier Transform

$$\hat{f}(y) = \int_G f(x) < x, y > dx.$$

Let $H \subset G$ be a closed subgroup. Define the orthogonal subgroup in G^* by

$$H^\perp = \{y \in G^* \mid < x, y >= 0 \ \ \forall x \in H\}.$$

Given Haar measures on G and H there exists a unique measure $dh^\perp$ on $H^\perp$ such that for all $f \in L^1(G)$ with $\hat{f} \in L^1(G^*)$ the *Poisson formula* holds:

$$\int_H f(x)dx = \int_{H^\perp} \hat{f}(y) dh^\perp.$$

Example 4.1. *Let $PL^+ \subseteq \mathbb{R}^{\ltimes} = (\sim_{\nVdash}, \ldots, \sim_{\ltimes})$ be the simplicial cone given by $s_j \geq 0$ for all $j = 1, \ldots, n$. Let N^1_{eff} be its $n-d$ dimensional image under a linear surjective map of vector spaces. We have an exact sequence*

$$0 \to M \to \mathbb{R}^{\ltimes} \to \mathbb{R}^{\ltimes -} \to \nvdash.$$

Then the function

$$\Lambda(N^1_{eff}, s) = \int_M \frac{dm}{\prod_{j=1}^n (s_j + \sqrt{-1} m_j)}$$

is a rational function on $N^1_{eff} \otimes \mathbb{R} = \mathbb{R}^{\ltimes -}$ *with simple poles on codimension 1 faces of* N^1_{eff}. *Here* m_j *are the coordinates of* m *in the fixed basis* $(s_1, ..., s_n)$ *of* $\mathbb{R}^{\ltimes}$. *This follows from Poisson formula applied to the Laplace transform of the characteristic function of the dual cone* $\widehat{N^1_{eff}}$.

4.3. **An integral representation.** Recall that for $Re(s) > 1$ we have defined the height zeta function by the series

$$Z_\Sigma(t,s) = \sum_{x \in T(F)} H_\Sigma(tx, -s).$$

In the following let us assume $t = 1$ and denote $Z_\Sigma(1,s)$ by $Z_\Sigma(s)$. The $\mathbf{K}$ invariance of the height H_Σ implies the following

Proposition 4.3.1. *Let* dx *be a Haar measure on* $T(\mathbb{A}_F)$. *There is an explicit constant* c_0 *such that*

$$Z_\Sigma(s) = c_0 \int_{\mathbf{KT(F)}} H_\Sigma(x, -s) dx.$$

Theorem 4.3.2. *The following formula holds for* $Re(s) > 1$

$$\int_{\mathbf{KT(F)}} H_\Sigma(x, -s) dx = \int_{(T(\mathbb{A}_F)/\mathbf{KT(F)})^*} dy \int_{T(\mathbb{A}_F)} H_\Sigma(x, -s) \chi_y(x) dx.$$

Here $\chi_y : T(\mathbb{A}_F)/\mathbf{KT(F)} \to \mathbb{C}$ *is a continuous character and* dx, dy *are appropriate Haar measures.*

4.4. **Reductions.** The character $\chi_y \in (T(\mathbb{A}_F)/\mathbf{KT(F)})$ splits through the section of the map $T^1(\mathbb{A}_F) \to T(\mathbb{A}_F)$ into a product

$$\chi_y = \chi_m \times \chi_u$$

such that χ_m is trivial on $T^1(\mathbb{A}_F)$ and $\chi_u \in (T^1(\mathbb{A}_F)/\mathbf{KT(F)})^*$. Notice that for split tori $T(\mathbb{A}_F)/T^1(\mathbb{A}_F) \simeq \mathbb{R} \simeq \mathbb{M}$ and that $\chi_{m,v} = |x|_v^{\sqrt{-1}m}$.

We can rewrite the integral above

$$Z_\Sigma(s) = \int_{(T(\mathbb{A}_F)/T^1(\mathbb{A}_F))^*} dm \int_{(T^1(\mathbb{A}_F)/\mathbf{KT(F)})^*} du \int_{T(\mathbb{A}_F)} H_\Sigma(x, -s) \chi_m(x) \chi_u(x) dx.$$

Consider

$$Z'_\Sigma(s) = \int_{(T(\mathbb{A}_F)/T^1(\mathbb{A}_F))^*} dm \int_{T(\mathbb{A}_F)} H_\Sigma(x, -s) \chi_m(x) dx.$$

A variant of Poisson formula implies that $Z_\Sigma(s)$ and $Z'_\Sigma(s)$ have the same analytic properties in the domain $Re(s) > 1 - \delta$ for some small positive δ. This is enough to prove the

Theorem 4.4.1. *Let* X_Σ *be a compactification of an anisotropic torus. Then the height zeta function* $Z_\Sigma(s)$ *satisfies Manin's conjecture (3.0.8).*

4.5. **Further reductions.** Just like $\mathbb{P}^{\nVdash}$ is seen to be a factor of $\mathbb{G}_{\eth}^{\nvDash}\setminus\{(\nvdash,\nvdash)\}$ by the diagonal action of $\mathbb{G}_{\gtrdot}$ one can represent split toric varieties as factors of some open dense $U(\Sigma)\hookrightarrow\mathbb{G}_{\eth}^{\ltimes}$ by an action of a torus $\mathbb{G}_{\gtrdot}^{\ltimes-}$. We have a diagramm

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{G}_{\gtrdot}^{\ltimes-} & \to & U(\Sigma) & \to & X_\Sigma & \to & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \to & \mathbb{G}_{\gtrdot}^{\ltimes-} & \to & \mathbb{G}_{\gtrdot}^{\ltimes} & \to & T & \to & 0 \end{array}$$

and an induced embedding of characters

$$T(\mathbb{A}_F)^*\hookrightarrow\mathbb{G}_{\gtrdot}^{\ltimes}(\mathbb{A}_{\mathbb{F}})^*.$$

Definition 4.5.1. *For* $x=(x_1,\dots,x_n)\in\mathbb{G}_{\gtrdot}^{\ltimes}(\mathbb{A}_{\mathbb{F}})$ *define* $f_\Sigma(x)=\otimes_v f_{\Sigma,v}(x)$ *by*

$$f_{\Sigma,v}(x)=\sum_{k=0}^{d}(-1)^{d-k}\sum_{\sigma\in\Sigma(k)}f_{\sigma,v}(x)$$

where $f_{\sigma,v}(x)=\prod_{j=1}^n f^j_{\sigma,v}(x)$ *and*

$$f^j_{\sigma,v}(x)=f^j_{\sigma,v}(x_j)=\{\begin{array}{lll} \delta_{1,v}(x_j) & if & \sigma\in\Sigma(d) \quad\quad and \quad e_j\notin\sigma \\ \chi_{1,v}(x_j) & if & \sigma\in\Sigma(k),\ k<d \quad and \quad e_j\notin\sigma \\ \chi_{\le 1,v}(x_j) & otherwise & \end{array}$$

Here $\delta_{1,v}(t)$ *is defined by*

$$\int_{\mathbb{G}_{\gtrdot}(\mathbb{F}_{\approx})}\delta_{1,v}(t)|x|^s\chi(t)dt=1$$

for all $\chi\in(T(\mathbb{A}_F)/\mathbf{K}T(F))^*$ *with* $Re(\chi)>0$ *and all* s *with* $Re(s)>0$. *The other functions are defined as*

$$\begin{array}{lllll} \chi_{1,v}(t)=1 & for & |t|_v=1 & and\ 0 & otherwise \\ \chi_{\le 1,v}(t)=1 & for & |t|_v\le 1 & and\ 0 & otherwise. \end{array}$$

The above definition needs some explanation. At each non-archimedian place $f_{\Sigma,v}(x)$ is a smooth function. At archimedian places $f_{\Sigma,v}(x)$ is a distribution. We don't want to define a product of delta functions. The notation $\delta_{1,v}(t_1)\times\delta_{1,v}(t_2)$ stands for $\delta_{(1,1)}(t_1,t_2)$.

Theorem 4.5.2. *For* $Re(s)>1$ *we have the following identity:*

$$\int_{T(\mathbb{A}_F)}H_\Sigma(x,-s)\chi_y(x)dx=\int_{\mathbb{G}_{\gtrdot}^{\ltimes}(\mathbb{A}_{\mathbb{F}})}f_\Sigma(x)|x|^s\chi_y(x)dx$$

Here χ_y *is considered as an element of* $(\mathbb{G}_{\gtrdot}^{\ltimes}(\mathbb{A}_{\mathbb{F}}))^*$ *and* $|x|^s=|x_1|^{s_1}\dots|x_n|^{s_n}$. *The integrals converge absolutely.*

Proof. To prove the theorem we compare local factors. Using

1. $T(\mathcal{O})_v$ invariance of $H_{\Sigma,v}$ and $\chi_{y,v}$,

2. the embedding $T(F_v)/T(\mathcal{O}_v) \hookrightarrow N$ (resp. $N \otimes \mathbb{R}$ for archimedian v),
3. the explicit description of the height
$$H_{\Sigma,v}(x_v) = e^{\varphi_s(\bar{x}_v)\log q_v}$$
where $\varphi_s \in PL(\Sigma)$ is a piecewise linear function on N (resp. $N \otimes \mathbb{R}$).

we deduce our claim. $\square$

We have expressed the height zeta function as a Mellin transform of a (generalized) function on a bigger group. The next step is to isolate the poles of the adelic integral
$$I_\Sigma(s, \chi_y) = \int_{\mathbb{G}_>^{\ltimes}(\mathbb{A}_\mathbb{F})} f_\Sigma(x)|x|^s \chi_y(x)dx,$$
and to study to integral
$$Z_\Sigma(s) = \int I_\Sigma(s, \chi_y)dy.$$

4.6. **Examples.** Assume $F = \mathbb{Q}$. In this section we consider split toric varieties. Recall that in this case $(T(\mathbb{A}_\mathbb{Q})/T^1(\mathbb{A}_\mathbb{Q}))^* \simeq \mathbb{R} \simeq \mathbb{M}$ and $\chi_m(x) = |x|^{\sqrt{-1}m}$. It follows that $I_\Sigma(s, \chi_m) = I_\Sigma(s + \sqrt{-1}m)$; our notations are in accordance with example (4.1). Let us introduce more notations: For $G = \prod \mathbb{G}_>$ define
$$J_\Sigma(s) = \int_{|x|\geq 1, x \in G(\mathbb{A}_F)} \hat{f}_\Sigma(x)|x|^{1-s}dx$$
where $\hat{f}_\Sigma(x)$ is the additive Fourier transform of $f_\Sigma(x)$. The integral converges uniformly for $Re(s) > 1 - \delta$ for some $\delta > 0$.

1. Here is yet another expression for the height zeta function of $\mathbb{P}^{\nvDash}$
$$Z_{\mathbb{P}^{\nvDash}}(s_1, s_2) =$$
$$\int_{\mathbb{R}^{\nvDash}} \frac{\zeta_\mathbb{Q}(s_1 + \sqrt{-1}m)\zeta_\mathbb{Q}(s_2 - \sqrt{-1}m)}{\zeta_\mathbb{Q}(s_1 + s_2)} \left(\frac{1}{(s_1 + \sqrt{-1}m)} + \frac{1}{(s_2 - \sqrt{-1}m)}\right) dm$$
where $\zeta_\mathbb{Q}(s)$ is the Riemann zeta function.
2. Consider a smooth split del Pezzo surface $X = X_\Sigma$ of degree 6. It is a toric variety obtained by blowing up $\mathbb{P}^{\nvDash}$ in 3 points or by compactifying $\mathbb{G}_>^{\nvDash}$ with 6 rational curves. The cone $N^1_{eff}(X)$ is generated by the divisors at infinity (exceptional curves) $D_1, \ldots, D_6$. From the discussion above we see that
$$Z_\Sigma(s) = \int_{\mathbb{R}^{\nvDash}} I_\Sigma(s + \sqrt{-1}m)dm$$
Put $s_{-1} = s_6$.

Claim.

$$I_\Sigma(s) = J_\Sigma(s) + \frac{c_\Sigma}{\prod_{j=1}^6 (s_j - 1)} +$$

$$+c_1 \sum_{j=1}^6 \frac{J_{\mathbb{P}^{\nVdash}}(s_{j+1}, s_{j-1})}{s_j(s_j - 1)} + c_2 \sum_{j=1}^6 \frac{1}{s_j s_{j+1}} +$$

$$+c_3 \sum_{j=1}^6 \frac{1}{(s_j - 1)(s_{j+1} - 1)} + c_4 \sum_{j=1}^6 \frac{1}{(s_j - 1) s_j (s_{j+1} - 1)}$$

$$-c_5 \sum_{j=1}^6 \frac{1}{s_j(s_{j+1} - 1)} - c_6 \sum_{j=1}^6 \frac{1}{s_j(s_{j-1} - 1)}.$$

Notice that

$$\int \frac{1}{\prod_{j=1}^6 (s_j - 1 + \sqrt{-1} m_j)} dm$$

coincides with the function

$$\Lambda(N^1_{eff}, s - 1) = \Lambda(-K_\Sigma + N^1_{eff}, s)$$

from example (4.1).

3. The constant c_Σ in the example (2) involves an Euler product. Let us give a general expression for a local factor:

$$c_{\Sigma,p} = \sum_{k=0}^d (-1)^{d-k} \sum_{\sigma \in \Sigma(k)} (1 - \frac{1}{p})^{n-k}.$$

From an easy combinatorial identity on the fan it follows that $1/p$ terms cancel and that the Euler product converges. For $\mathbb{P}^{\nVdash}$ we get $\frac{1}{\zeta(2)}$ as expected, for $\mathbb{P}^{\nvDash}$ blown up in 3 points we get

$$c_{\Sigma,p} = (1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6}).$$

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Universität-GHS-Essen, Fachbereich 6, Mathematik Universitätsstr. 3,, 45141 Essen, Federal Republic of Germany
E-mail address: matf∅∅@vm.hrz.uni-essen.de

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: tschink@math.harvard.edu