MODELS OF TRIPLE COVERS

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ABSTRACT. We exhibit, for a degree 3 covering of algebraic varieties, a model where the covering is a finite covering of smooth projective varieties branched over a smooth divisor.

1. INTRODUCTION

Let k be a field of characteristic 0. It is well known that any morphism of projective varieties $\psi\colon T \to S$ over k, that is generically finite of degree 2, can be put into a commutative diagram

$$\begin{array}{ccc} \widetilde{T} & \overset{\varrho_T}{\dashrightarrow} & T \\ {\scriptstyle\tilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{S} & \xrightarrow{\varrho_S} & S \end{array} \tag{1.1}$$

with smooth projective varieties $\widetilde{S}$ and $\widetilde{T}$, such that ϱ_S is a birational morphism, ϱ_T is a birational map, and $\tilde{\psi}$ is a degree 2 finite covering. The branch locus of a degree 2 finite covering of smooth projective varieties is a smooth divisor.

In this note, we establish an analogous theorem for triple covers, over perfect fields of characteristic not equal to 2 or 3. Good models of triple covers are important for the construction of models of fibrations (over a base of arbitrary dimension), when the symmetry group of the geometric generic fiber admits the symmetric group $\mathfrak{S}_3$ as a quotient, as is the case for fibrations in sextic del Pezzo surfaces [3] [2] [1] [6].

There is an extensive literature on triple covers of surfaces, e.g., [4] [7] [11] [10] [9]. In [10], a theorem similar to the one in this note is proved for triple covers of surfaces, by a method related to the classical solution of a cubic equation. Our approach is more geometric, and is based on an analysis of ramification in codimension 1 and 2.

We produce $\tilde{\psi}\colon \widetilde{T} \to \widetilde{S}$, ramified over a smooth divisor of $\widetilde{S}$. By contrast, many of the (non-cyclic) degree 3 coverings of smooth projective varieties that occur naturally have singular branch locus, such as 6-cuspidal sextic with cusps on a conic as branch locus of a general

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projection of a smooth cubic surface [12]. Such a covering has total ramification (points of S with, geometrically, just one pre-image in T) at the cusps and simple ramification (two pre-images) at smooth points of the branch locus; after the procedure described here has been applied the loci with total and simple ramification comprise disjoint divisors.

As explained in [8, Exa. 3.1], models of degree ≥ 4 covers of surfaces as in (1.1) do not exist in general.

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2. Finite covers and ramification

Let k be a field and S a smooth variety over k, i.e., a separated geometrically integral scheme of finite type over k. By the normalization of S in a finite field extension of $k(S)$ we have a canonical correspondence between finite field extensions of $k(S)$ and connected normal k-schemes with finite surjective morphism to S. The set of points where such a morphism $T \to S$ fails to be étale is a closed subset $Z \subset T$ which is

- equal to T if and only the associated finite field extension of $k(S)$ is inseparable;
- is otherwise of pure codimension 1 or empty, by the Zariski-Nagata purity theorem [5, Thm. X.3.1].

Consequently, if there exists a simple normal crossing divisor $D \subset S$, $D = D_1 \cup \cdots \cup D_n$ with D_i irreducible for all i, such that $T \times_S (S \smallsetminus D) \to S$ is étale, then the branch locus (the image of Z in S) is of the form $\bigcup_{i \in I} D_i$ for some $I \subset \{1, \ldots, n\}$.

Once a finite field extension of $k(S)$ has been specified, when we refer to the branch locus we mean the branch locus of $T \to S$, where T is the normalization of S in the given field extension of $k(S)$.

3. Main result

Let k be a perfect field of characteristic not equal to 2 or 3, S a smooth projective variety over k, and T a projective variety with morphism to S that is generically finite of degree 3. If the field extension $k(T)/k(S)$ is cyclic, then an argument just as in the case of double covers yields a commutative diagram (1.1), where $\widetilde{\psi} \colon \widetilde{T} \to \widetilde{S}$ is a cyclic degree 3 covering of smooth projective varieties branched over a smooth divisor on $\widetilde{S}$. As in the case of double covers, a form of resolution of singularities for divisors on S is required; for instance, it is sufficient if the following is available:

(R) Embedded resolution of singularities of divisors on S by iterated blow-up with smooth center.

Of course, this is available (Hironkaka) when k has characteristic zero. When k has positive characteristic, this is available for $\dim(S) \leq 3$ (trivial/classical for $\dim(S) \leq 2$, due to Abhyankar for $\dim(S) = 3$).

Theorem 1. *Let k be a perfect field of characteristic not equal to 2 or 3, S a smooth projective variety over k, and $T \to S$ a morphism of projective varieties that is generically finite of degree 3. We suppose that* (R) *holds for S. Then there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{T} & \overset{\varrho_T}{\dashrightarrow} & T \\ {\scriptstyle \tilde{\psi}} \downarrow & & \downarrow {\scriptstyle \psi} \\ \widetilde{S} & \xrightarrow{\varrho_S} & S \end{array}$$

with smooth projective varieties $\widetilde{S}$ and $\widetilde{T}$, such that

- *ϱ_S is a birational morphism,*
- *ϱ_T is a birational map, and*
- *$\tilde{\psi}$ is a degree 3 finite covering, branched over a smooth divisor of $\widetilde{S}$.*

Proof. As mentioned, when $k(T)/k(S)$ is cyclic this is achieved by applying (R) to make the branch locus into a strict normal crossing divisor and repeatedly blowing up components of the intersection of a pair of components of the branch locus to make the branch locus smooth. Essential for the second step is the observation that a component Y of an intersection $D_i \cap D_j$ $(i \neq j)$ of components of the strict normal crossing branch locus $D = D_1 \cup \cdots \cup D_n$ (achieved by the first step) may be assigned to one of two types, according to whether the branch locus for $k(T)/k(S)$ of the blow-up $B\ell_Y S$ has the exceptional divisor as a component. Let ℓ, respectively m denote the number of components $Y \subset D_i \cap D_j$ for some $i \neq j$, for which the exceptional divisor of $B\ell_Y S$ is, respectively is not, a component of the branch locus. (For simplicity of notation we continue to write S, even though the first step potentially makes a birational modification.) By blowing up some Y for which the exceptional divisor is not a component of the branch locus whenever this is possible, and otherwise blowing up any Y, we obtain for $(\ell, m) \neq (0,0)$ a pair (ℓ', m') associated with $S' := B\ell_Y S$ that is smaller than (ℓ, m) in lexicographic order.

In the non-cyclic case we proceed with the same first step, making the branch locus into a simple normal crossing by applying (R). Now the branch locus has components of two kinds: some, say $D_1 \cup \cdots \cup D_n$, with simple ramification and others, $D'_1 \cup \cdots \cup D'_{n'}$ with total ramification. We adopt the convention that the D_i and $D'_{i'}$ are all irreducible.

The discriminant of $k(T)/k(S)$ determines a quadratic extension of $k(S)$ with branch locus $D_1 \cup \cdots \cup D_n$. By blowing up intersections of pairs of components, we achieve $D_i \cap D_j = \emptyset$ for $i \neq j$.

We claim that $D_i \cap D'_{i'} = \emptyset$ for all i and i'. This follows from the fact that the pre-image of $D'_{i'}$ in the discriminant double cover is irreducible. We argue by contradiction, replacing S by a strict henselization of the local ring of S at the generic point of a component of $D_i \cap D'_{i'}$. We still have a cubic extension of the residue field of the generic point, still with nontrivial discriminant, and this yields a cyclic cubic extension of the discriminant cover. This must be obtained from a cyclic cubic extension below by base change to the double cover, and we have a contradiction.

We may have $D'_{i'} \cap D'_{j'} \neq \emptyset$ for some $i' \neq j'$, but then as described at the beginning of the proof we may deal with this by blowing up components of intersections $D'_{i'} \cap D'_{j'}$. Then we have a smooth branch locus and hence a smooth model of the covering. □

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