SPECIALIZATION OF BIRATIONAL TYPES

MAXIM KONTSEVICH AND YURI TSCHINKEL

1. Introduction

This paper is inspired by the discovery by Larsen and Lunts [LL03] of a remarkable connection between motivic integration and stable rationality and by the recent development of these ideas by Nicaise and Shinder [NS17], who proved that stable rationality is preserved under specializations in smooth families. Our goal here is to simplify and strengthen their arguments, leading to the following:

Theorem 1. *Let*

$$\pi : \mathcal{X} \to B \quad \textit{and} \quad \pi' : \mathcal{X}' \to B$$

be smooth proper morphisms to a smooth, connected curve B, over a field of characteristic zero. Assume that the generic fibers of π and π' are birational over the function field of B. Then, for every closed point $b \in B$, the fibers of π and π' over b are birational over the residue field at b.

In particular, if the generic fiber of π is rational then every fiber of π is rational.

Specialization of rationality was known for families of relative dimension 3 [Tim82], [dFF13]. In fact, in Section 4 we prove a stronger specialization result, when the special fiber has singularities of a certain type, e.g., rational double points. An immediate consequence of our theorem is that for every smooth proper family over a higher-dimensional base the locus of points on the base where the fibers are geometrically rational is a countable union of closed subsets.

Throughout, we work over a field k of characteristic zero. In our approach, we replace the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$ of varieties over k, which plays a key role in motivic integration and which is crucial in [NS17], by a new invariant, the *Burnside ring* of k, denoted by

$$\mathrm{Burn}(k).$$

Date: August 14, 2017.

Its definition and formal properties resemble those of the classical notion of the Burnside ring of a finite group (see Section 2).

This ring seems better adapted to questions of rationality: while $\mathrm{K}_0(\mathrm{Var}_k)$, and its quotient modulo the ideal generated by the class of the affine line, are natural in the context of *stable* rationality, the ring $\mathrm{Burn}(k)$ captures directly birational types: as an abelian group, $\mathrm{Burn}(k)$ is freely generated by isomorphism classes of function fields over k. It is naturally graded, by the transcendence degree, and admits a surjection

$$\mathrm{Burn}(k) \to \mathrm{gr}(\mathrm{K}_0(\mathrm{Var}_k)),$$

onto the associated graded, with respect to the dimension filtration. In the early days of motivic integration, it was believed that this map is an isomorphism, but now it is known that the kernel is nontrivial [Bor14, Theorem 2.13].

The proof of Theorem 1 is based on the existence of a *specialization morphism* defined in Section 3. This specialization morphism is additive but does not preserve the multiplicative structure of Burnside rings. In Section 4, we introduce Burnside *groups* for schemes and define a class of singularities relevant for rationality considerations. We then prove a specialization of rationality theorem in this context. In Section 5, we refine the notion of Burnside rings which insures the multiplicativity of the specialization map in this more general situation.

Acknowledgments: The second author was partially supported by NSF grant 1601912.

2. Burnside rings of fields

We recall the following classical notion. Let G be a finite group. Denote by $\mathrm{Burn}_+(G)$ the set of isomorphism classes of finite G-sets; it is a commutative semi-ring with addition and multiplication given by disjoint union and Cartesian product. As an additive monoid, $\mathrm{Burn}_+(G)$ is a free commutative monoid generated by the set $\mathrm{Conj}(G)$ of conjugacy classes of subgroups of G. The Burnside ring

$$\mathrm{Burn}(G)$$

is defined as the associated Grothendieck ring. It is a free $\mathbb{Z}$-module generated by $\mathrm{Conj}(G)$. The ring $\mathrm{Burn}(G)$ is a combinatorial avatar of the representation ring $R_F(G)$, which is the Grothendieck ring of finite dimensional G-representations over a field F: there is a canonical ring homomorphism

$$\mathrm{Burn}(G) \to R_F(G)$$

for any field F.

This definition generalizes to pro-finite groups. In particular, we can apply this to Galois groups G_k of fields k. In this case, the set $\mathrm{Conj}(G_k)$ is the set of isomorphism classes of finite field extensions L/k, which we can interpret as the set of isomorphism classes of zero-dimension k-schemes which are non-empty, connected, and reduced. The corresponding semi-ring $\mathrm{Burn}_+(G_k)$ can be described as the set of equivalence classes of all smooth zero-dimensional (not necessarily connected) schemes over k.

We propose the following generalization of these notions: Let $\sim_k$ be the equivalence relation on smooth, but possibly not connected and not equidimensional, schemes over k generated by open embeddings with dense image.

Definition 2. The Burnside semi-ring $\mathrm{Burn}_+(k)$ of a field k is the set of $\sim_k$-equivalence classes of smooth schemes over k of finite type endowed with a semi-ring structure where multiplication and addition are given by disjoint union and product over k.

For a smooth scheme S/k we denote by $[S/k]$ the corresponding $\sim_k$-equivalence class. As an additive monoid, $\mathrm{Burn}_+(k)$ is freely generated by the set of $\sim_k$-equivalence classes of smooth nonempty connected schemes; for two such schemes X, X' we have $[X/k] = [X'/k]$ if and only if X and X' are k-birational. Therefore, we can identify additive generators of $\mathrm{Burn}_+(k)$ with $\sqcup_{n\geq 0}\mathrm{Bir}_n(k)$, where

$$\mathrm{Bir}_n(k), \quad n \in \mathbb{Z}_{\geq 0}$$

is the set of equivalence classes of irreducible algebraic varieties over k of dimension n, modulo k-birational equivalence. Abusing notation, we will write $[L/k]$ the class of any smooth variety X with function field $L = k(X)$. Note that

$$\mathrm{Bir}_0(k) = \mathrm{Conj}(G_k).$$

We denote by

$$\mathrm{Burn}(k)$$

the Grothendieck ring generated by $\mathrm{Burn}_+(k)$. Clearly, $\mathrm{Burn}(k)$ carries a natural grading by the transcendence degree over k,

$$\mathrm{Burn}(k) = \oplus_{n\geq 0}\mathrm{Burn}_n(k)$$

and

$$\mathrm{Burn}_0(k) = \mathrm{Burn}(G_k).$$

Remark 3. This construction provides a functor from the category of fields of characteristic zero, with morphisms given by inclusion of fields, to the category of $\mathbb{Z}_{\geq 0}$-graded commutative rings.

3. Specialization

Let $\mathfrak{o}$ be a complete discrete valuation ring with residue field k and fraction field K. Since $\mathrm{char}(k) = 0$, by our assumptions, we have non-canonical isomorphisms $\mathfrak{o} \simeq k[[t]]$ and $K \simeq k((t))$. Our goal is to define a specialization homomorphism of graded *abelian groups* (preserving addition, but not multiplication):

$$\rho : \mathrm{Burn}(K) \to \mathrm{Burn}(k). \tag{3.1}$$

This requires a collection of maps

$$\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}[\mathrm{Bir}_n(k)],$$

for all $n \in \mathbb{Z}_{\geq 0}$. Here $\mathbb{Z}[\mathrm{Bir}_n(k)]$ is the free abelian group, generated by classes in $\mathrm{Bir}_n(k)$. From these, we will obtain ρ by $\mathbb{Z}$-linearity.

To define $\rho_n([L/K])$ we proceed in two steps:

(1) Choose a smooth proper (or projective) variety X/K of dimension n with function field $L = K(X)$.
(2) Choose a regular model of X

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o})$$

such that π is proper and the special fiber $\mathcal{X}_0$ over $\mathrm{Spec}(k)$ is a simple normal crossings (snc) divisor.

The last condition means that

$$\mathcal{X}_0 = \cup_{i \in I} d_i D_i,$$

where I is a finite set, $d_i \in \mathbb{Z}_{\geq 1}$ and D_i are smooth irreducible divisors in $\mathcal{X}$, with transversal intersections. We put

$$\rho_n([L/K]) := \sum_{\emptyset \neq J \subseteq I} (-1)^{\#J-1} [D_J \times \mathbb{A}^{\#J-1}/k], \tag{3.2}$$

where

$$D_J := \cap_{j \in J} D_j. \tag{3.3}$$

This formula is inspired by the formula (3.2.2) in [NS17]. Note that this map ρ is a *not* a ring homomorphism; a modified version which *is* a ring homomorphism will be described in Section 5.

Theorem 4. *The maps*

$$\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}[\mathrm{Bir}_n(k)], \quad n \geq 0,$$

given by (3.2)*, are well-defined.*

Proof. We need to establish the following:

(1) For a given smooth proper X over K, the right side of (3.2) does not depend on the choice of a model $\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o})$.
(2) Assuming this independence of the model $\mathcal{X}$, the right side of (3.2) does not depend on the choice of a smooth proper model X/K of the field extension L/K.

The proofs of properties (1) and (2) are similar to the arguments in [Bit04, Section 3]. We start with (1). By the Weak Factorization Theorem for birational maps between smooth proper k-varieties [Wło03], [AKMW02], it suffices to check that the right side of (3.2) does not change under the following elementary transformation: the blowup

$$\beta : \tilde{\mathcal{X}} := \mathrm{Bl}_Z(\mathcal{X}),$$

where Z is a smooth closed irreducible subvariety in D_{J_0}, for some $\emptyset \neq J_0 \subseteq I$, and such that

- $\dim(Z) \leq \dim(X) - 2$,
- Z intersects the divisors D_i transversally and
- for $i \notin J_0$, the divisors $Z_i := D_i \cap Z$ of Z are all distinct and form a normal crossings divisor in Z.

The k-irreducible components of the new special fiber $\tilde{\mathcal{X}}_0$ are labeled by $\tilde{I} := I \sqcup \{i_Z\}$, where the new component corresponding to i_Z is the exceptional divisor of β. We let $\iota : I \hookrightarrow \tilde{I}$ be the natural inclusion map. There are two cases:

(a) $\dim(Z) < \dim(D_{J_0})$,
(b) Z is a component of D_{J_0}.

We start with (a). For $J \subseteq I$, the stratum $D_{\iota(J)} \subset \tilde{\mathcal{X}}_0$ is a blowup of D_J, and thus birationally equivalent to D_J. Hence, we already matched those terms of formula (3.2), whose indices do not contain i_Z.

The terms which do contain i_Z are labelled by the following data:

- a subset $J_1 \subseteq I \setminus J_0$,
- an element $\alpha \in \pi_0(Z \cap D_{J_1})$, (the set of connected components).
- a subset $J_0' \subseteq J_0$.

The corresponding subset in $\tilde{I}$ is $\tilde{J} := \iota(J_0' \cup J_1) \cup \{i_Z\}$. Note that

$$\pi_0(D_{\tilde{J}}) = \pi_0(Z \cap D_{J_1})$$

and thus we can use $\alpha \in \pi_0(Z \cap D_{J_1})$ as a label for a component $D_{\tilde{J},\alpha}$ of $D_{\tilde{J}}$. Moreover, we have a k-birational equivalence

$$D_{\tilde{J},\alpha} \times \mathbb{A}^{\mathrm{codim}_{\tilde{\mathcal{X}}}(D_{\tilde{J},\alpha})-1} \sim_k (Z \cap D_{J_1})_\alpha \times \mathbb{A}^{\mathrm{codim}_{\mathcal{X}}(Z \cap D_{J_1})_\alpha - 1}.$$

Observe that, for given J_1 and α, the alternating sum over J_0' vanishes. It follows that the sum over all terms containing i_Z is zero. This proves the Case (a) of (1).

We turn to Case (b). Then, by assumptions, $\#J_0 \geq 2$. For simplicity of the exposition, we may assume that D_{J_0} is connected and hence, $Z = D_{J_0}$. The nonempty intersections of divisors in $\tilde{X}$ are of the form

- $D_{\iota(J)} \sim_k D_J$, for $\emptyset \neq J \subseteq I, I_0 \not\subseteq J$,
- $D_{\iota(J)\cup\{i_Z\}} \sim_k D_{J\cup I_0} \times \mathbb{A}^{\#(J\cup I_0)-\#J-1}$, for J such that $I_0 \not\subseteq J$.

A direct calculation shows that the right side of (3.2) does not change.

To show (2), we use the Weak Factorization Theorem to reduce the claim to the case of a blowup

$$\beta : \tilde{X} := \mathrm{Bl}_Y(X) \to X$$

with smooth center $Y \subset X$. Then, using embedded resolution of singularities compatible with a divisor, we can find a model $\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o})$ with special fiber a divisor with simple normal crossings $\cup_{i\in I} d_i D_i$ such that the closure $\mathcal{Y}$ of Y in $\mathcal{X}$ is smooth and $D_i \cap \mathcal{Y}$ are simple normal crossings divisors in $\mathcal{Y}$ (see e.g., [Bit04, Section 3]). Note that the set of irreducible components of the special fiber $\mathcal{X}_0$ does not change under β and the corresponding components $D_{J,\alpha}$ are replaced by their proper transforms in the blowup; this preserves their birational equivalence type. This proves (2). □

We are now in the position to deduce Theorem 1.

Proof. Let $\pi : \mathcal{X} \to B$ be a smooth proper morphism to a smooth connected curve B over k with fiber X over the generic point of B. Let $K = k(B)$ be the function field of B. Let κ_b be the residue field at b, a finite extension of k. Let K_b be the completion of K at b. It is a local field with residue field κ_b, isomorphic to $\kappa_b((t))$, where t is a formal local coordinate. Let

$$\phi_b : K \to K_b$$

be the canonical inclusion. By functoriality (see Remark 3), it defines a homomorpism

$$\phi_{b,*} : \mathrm{Burn}(K) \to \mathrm{Burn}(K_b).$$

We have the specialization homomorphism

$$\rho : \mathrm{Burn}(K_b) \to \mathrm{Burn}(\kappa_b)$$

and the following identity

$$[X_b/\kappa_b] = \rho(\phi_{b,*}([X/K])),$$

which follows immediately from the definition of ρ, since the special fiber is smooth and irreducible. This shows that the birational type of the fiber is determined by the birational type at the generic point. □

Remark 5. A variety X/k is called stably rational of level $\leq r$ if $X \times \mathbb{A}^r$ is k-rational. An immediate corollary is that if the generic fiber of a family $\pi : \mathcal{X} \to B$, as in Theorem 1, is stably rational of level $\leq r$ then every fiber is stably rational of level $\leq r$.

Remark 6. In the notation of Theorem 1, given a birational isomorphism of generic fibers of π and π', the corresponding birational isomorphism of the special fibers is not determined uniquely. However, following the proof of Theorem 4, we can determine an explicit chain of elementary birational modifications relating the special fibers $\mathcal{X}_b$ and $\mathcal{X}'_b$, for a given $b \in B$.

Definition 7. A variety X/k is called (birationally) $\mathbb{A}^1$-divisible, if there exists a variety Y/k such that $[X/k] = [Y \times \mathbb{A}^1/k]$.

Remark 8. In the notation of Theorem 1, if the generic fiber of π is $\mathbb{A}^1$-divisible then so is the special fiber.

Remark 9. Assume that for some $b \in B$, the fiber $\mathcal{X}_b$ is not $\mathbb{A}^1$-divisible. The considerations above show that there is a canonical restriction homomorphism between groups of birational automorphisms

$$\mathrm{BirAut}(X/K) \to \mathrm{BirAut}(\mathcal{X}_b/\kappa_b). \tag{3.4}$$

Indeed, using formula (3.1) and following the steps of the proof of Theorem 4, we see that for every model $\pi : \mathcal{X} \to B$ of $L = K(X)$ and every local model $\mathcal{X}'_b \to \mathrm{Spec}(\mathfrak{o})$, every summand, but one, in the formula (3.2) for the image of the specialization map ρ, is $\mathbb{A}^1$-divisible. This allows to define the homomorphism (3.4).

4. Burnside groups for schemes

Definition 10. Let S/k be a separated scheme of finite type over k. The Burnside monoid $\mathrm{Burn}_+(S/k)$ is the set of equivalence classes of maps $f : X \to S$, where X is smooth over k, modulo the equivalence relation generated by

$$(X, f) \sim_k (U, f|_U),$$

where $U \hookrightarrow X$ is an open embedding with dense image. The monoid structure on $\mathrm{Burn}_+(S/k)$ is given by disjoint union.

We write $[X \xrightarrow{f} S]$ for the equivalence class defined above. We denote by

$$\mathrm{Burn}(S/k)$$

the corresponding Grothendieck group (it is no longer a ring). As before, the Burnside monoid and group are naturally graded by the dimension of X over k. We recover Definition 2 when $S = \mathrm{Spec}(k)$.

Note that $\mathrm{Burn}_{+,n}(S/k)$, for $n \in \mathbb{Z}_{\geq 0}$, is freely generated by the set

$$\coprod_{s \in S,\, \dim(s) \leq n} \mathrm{Bir}_{n-\dim(s)}(\kappa_s),$$

where κ_s is the residue field of the Zariski point $s \in S$.

Functoriality: A morphism of schemes of finite type $g : S' \to S$ over k induces a homomorphism of Burnside groups

$$g_* : \mathrm{Burn}(S'/k) \to \mathrm{Burn}(S/k), \quad g_*([X \xrightarrow{f} S']) := [X \xrightarrow{g \circ f} S],$$

preserving the grading.

Let X/k be a reduced, separated, equidimensional algebraic variety, possibly reducible, nonproper, or singular. Let $Z \subset X$ be a closed subvariety of dimension $< \dim(X)$. To such a pair (X, Z) we will associate an element

$$\partial_Z(X) \in \mathrm{Burn}_{\dim(X)-1}(Z/k).$$

These assigments will satisfy the following conditions

(1) If X is smooth and $Z = \cup_{i \in I} D_i$ is an snc divisor in X then

$$\partial_Z(X) = \sum_{\emptyset \neq J \subseteq I} (-1)^{\#J-1} [D_J \times \mathbb{A}^{\#J-1} \xrightarrow{f_J} Z], \tag{4.1}$$

where f_J is the composition of projection to the first factor with the natural inclusion $D_J \hookrightarrow Z$.

(2) Let $g : X' \to X$ be a proper surjective morphism and $Z' := g^{-1}(Z)$. Assume that g induces a birational isomorphism between X' and X (in particular, $\dim(X') = \dim(X)$), and (X', Z') satisfies the conditions above. Then

$$\partial_Z(X) = (g|_Z)_*(\partial_{Z'}(X')). \tag{4.2}$$

Theorem 11. *The invariants $\partial_Z(X)$ satisfying (1) and (2) exist and are uniquely defined.*

Proof. The properties (1) and (2) provide a definition, using resolution of singularities. The proof of independence on the choice of such a resolution is parallel to the proof of property (1) in Theorem 4. □

Definition 12. Let X/k be an irreducible algebraic variety and $Z \subset X$ an irreducible divisor. The pair (X, Z) has *B-rational singularities* if

$$\partial_Z(X) = [Z^{\text{smooth}} \hookrightarrow Z].$$

Note that the difference

$$\partial_Z(X) - [Z^{\text{smooth}} \hookrightarrow Z]$$

is always supported in $Z^{\text{sing}} \cup (X^{\text{sing}} \cap Z)$. In particular, if Z and X are both smooth, then the pair (X, Z) trivially has B-rational singularities.

Example 13. Let X/k be a smooth variety and $Z \subset X$ a divisor, defined over k, with an isolated singularity at $z \in Z(k)$. Let $\beta : \tilde{X} := \mathrm{Bl}_z(X)$ be the blowup. Assume that

- the proper transform Z' of Z is smooth and intersects the exceptional divisor $E \simeq \mathbb{P}^{\dim(X)-1}$ transversally,
- the cone over $E \cap Z'$ is k-rational.

Then the pair (X, Z) has B-rational singularities.

Indeed, on $\tilde{X}$ we have the divisor $\tilde{Z} := \beta^{-1}(Z) = Z \cup E$. The formula (4.1) for $\partial_{\tilde{Z}}(\tilde{X})$ has three terms:

$$[E \to \tilde{Z}] + [Z' \to \tilde{Z}] - [(E \cap Z') \times \mathbb{A}^1 \to \tilde{Z}]$$

We apply (4.2) to the blowup β and find

$$\partial_Z(X) := [\mathbb{A}^{\dim(X)-1} \to z] + [Z^{\text{smooth}} \hookrightarrow Z] - [(E \cap Z') \times \mathbb{A}^1 \to z].$$

By our assumption, we find that the first and the third terms cancel. Hence

$$\partial_Z(X) := [Z^{\text{smooth}} \hookrightarrow Z],$$

which is the definition of B-rationality.

Example 14 (a special case of the previous example)**.** Consider pairs (X, Z) with smooth X/k. Assume that Z is a divisor that has only ordinary double point singularities at k-rational points, and such that the associated projective quadrics all have k-rational points. Then the pair (X, Z) has B-rational singularities.

Example 15. Consider pairs (X, Z), with X/k a smooth surface (e.g., $\mathbb{A}^2$). Assume that $Z \subset X$ is a closed irreducible curve over k which is unibranched at every singular point. Then (X, Z) has B-rational singularities.

Theorem 16. *Let $\pi : \mathcal{X} \to B$ be a proper flat morphism from a possibly singular irreducible variety to a smooth connected curve B over k. Assume that the generic fiber X is geometrically connected over $K := k(B)$. Let $b \subset B$ be a closed point and $\mathcal{X}_b := \pi^{-1}(b)$ the fiber over b. Assume that the pair $(\mathcal{X}, \mathcal{X}_b)$ has B-rational singularities. If X is rational over K then $\mathcal{X}_b$ is rational over the residue field κ_b at b.*

Proof. Let $\beta : \tilde{\mathcal{X}} \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$ which is an isomorphism on the complement to $\mathcal{X}_b \cup \mathcal{X}^{\mathrm{sing}}$ and such that $\beta^{-1}(\mathcal{X}_b \cup \mathcal{X}^{\mathrm{sing}})$ is an snc divisor in $\tilde{\mathcal{X}}$. By our assumption we have that $\tilde{\mathcal{X}}_b = \beta^{-1}(\mathcal{X}_b)$ is a connected (but possibly reducible) component of $\beta^{-1}(\mathcal{X}_b \cup \mathcal{X}^{\mathrm{sing}})$. We have

$$\beta_*(\partial_{\tilde{\mathcal{X}}_b}(\tilde{\mathcal{X}})) = \partial_{\mathcal{X}_b}(\mathcal{X})$$

By the assumption that the singularities of the pair $(\mathcal{X}, \mathcal{X}_b)$ are B-rational, we also have

$$\partial_{\mathcal{X}_b}(\mathcal{X}) = [\mathcal{X}_b^{\mathrm{smooth}} \hookrightarrow \mathcal{X}_b]. \tag{4.3}$$

We have the natural projections

$$\Pi_b := \pi|_{\mathcal{X}_b} \to b, \quad \tilde{\Pi}_b : \tilde{\pi}|_{\tilde{\mathcal{X}}_b} \to b, \tag{4.4}$$

(we identify $b = \mathrm{Spec}(\kappa_b)$), and

$$\tilde{\Pi}_b = \Pi_b \circ \beta|_{\tilde{\mathcal{X}}_b}, \quad \tilde{\pi} = \pi \circ \beta.$$

Apply the induced homomorphism $(\Pi_b)_*$ to both sides of (4.3). Using functoriality, we obtain that

$$(\tilde{\Pi}_b)_*(\partial_{\tilde{\mathcal{X}}_b}(\tilde{\mathcal{X}})) = [\mathcal{X}_b^{\mathrm{smooth}} \to b] \tag{4.5}$$

Note that both sides of (4.5) are elements of $\mathrm{Burn}(b/\kappa_b) = \mathrm{Burn}(\kappa_b)$. The left side of (4.5) is, by definition,

$$\rho(\phi_{b,*}[X/K]),$$

where ρ is the specialization homomorphism defined in Section 3 and $\phi_b : K \to K_b$ is the natural embedding into the completion at b.

Rationality of X over K implies that $[X/K] = [\mathbb{A}^{\dim(X)}/K]$, so that the left side of (4.5) equals

$$[\mathbb{A}^{\dim(X)} \to b] = [\mathbb{A}^{\dim(X)}/\kappa_b].$$

Hence $\mathcal{X}_b$ is rational. □

Remark 17. Recently, there has been rapid progress in establishing failure of (stable) rationality via specialization to varieties with computable obstructions to (stable) rationality, see, e.g., [Voi15], [CTP16], [Tot16], [HKT16]. Previously, specialization was studied on the level of integral decomposition of the diagonal [Voi15], universal CH_0-triviality [CTP16], or $\mathrm{K}_0(\mathrm{Var}_k)$ [NS17]. We expect that the flexibility of Theorem 16 will lead to new applications in this area.

5. Refined specialization

We keep the notation of Section 3: we consider a regular model

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o}),$$

with special fiber

$$\mathcal{X}_0 := \cup_{i \in I} d_i D_i.$$

We defined an additive homomorphism

$$\rho : \mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

where K is the fraction field of $\mathfrak{o}$ and k its residue field. Our first goal here is to decompose

$$\rho := \sum_{d \geq 1} \rho^{(d)},$$

where

$$\rho^{(d)} := \sum_{n \geq 0} \rho_n^{(d)}, \quad \rho_n^{(d)} : \mathrm{Bir}_n(K) \to \mathbb{Z}[\mathrm{Bir}_n(k)].$$

The presentation as an infinite sum is meaningful, since for any given element in $\mathrm{Burn}(K)$, the evaluation of $\rho^{(d)}$ vanishes for sufficiently large d. Put

$$\rho_n^{(d)}([L/K]) := \sum_{\emptyset \neq J \subseteq I, d_J = d} (-1)^{\#J-1} [D_J \times \mathbb{A}^{\#J-1}/k], \tag{5.1}$$

where

$$d_J := \gcd(d_j)_{j \in J}.$$

The proof that $\rho_n^{(d)}$ is well-defined is identical to the proof of Theorem 4, one has only to keep track of the multiplicities d_J on the blowup.

We can organize the additive homomorphisms into a generating series

$$\hat{\rho} := \sum_{d\geq 1} \rho^{(d)} T^d : \mathrm{Burn}(K) \to \mathrm{Burn}(k)[T], \tag{5.2}$$

where T is a formal variable; evaluating at $T = 1$ we obtain ρ.

Our ultimate goal is to define a specialization homomorphism

$$\rho_\tau^{\boldsymbol{\mu}} : \mathrm{Burn}(K) \to \mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k),$$

to an equivariant version of the Burnside ring defined below. The homomorphism $\rho_\tau^{\boldsymbol{\mu}}$ will be compatible with multiplication.

Step 1. Let

$$\hat{\boldsymbol{\mu}} := \varprojlim \boldsymbol{\mu}_d,$$

where $\boldsymbol{\mu}_d$ are k-group schemes of d-th roots of 1. We define rings

$$\mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k)$$

in same manner as $\mathrm{Burn}(k)$, except that we now consider smooth k-schemes, together with a continuous $\hat{\boldsymbol{\mu}}$-action. The multiplication is given by the product of schemes. The additive generators of this Grothendieck group are equivalence classes $[\bar{X} \to X, d]$, where $d \in \mathbb{Z}_{\geq 1}$, and $\bar{X} \to X$ is a $\boldsymbol{\mu}_d$-torsor over a smooth scheme X. In terms of fields, the generators are isomorphism classes $[L/k, \lambda, d]$, where L/k is a function field, $d \in \mathbb{Z}_{\geq 1}$, and $\lambda \in L^\times/(L^\times)^d$ (by Kummer theory).

Step 2. Fix a nonzero element

$$\tau \in \mathfrak{m}/\mathfrak{m}^2 \simeq k,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{o}$. The homomorphism $\rho_\tau^{\boldsymbol{\mu}}$ depends on this choice.

Step 3. As in Section 3, we assume that the k-irreducible components D_i of the special fiber $\mathcal{X}_0$ form a strict normal crossings divisor. Denote by $\mathcal{N}_i$ the normal bundle of D_i. The fibration structure $\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o})$ defines, for all $\emptyset \neq J \subseteq I$, an isomorphism

$$\nu'_J : \bigotimes_{j\in J} \left(\mathcal{N}_{j|_{D_J^\circ}}\right)^{\otimes d_j} \xrightarrow{\sim} \mathfrak{m}/\mathfrak{m}^2 \otimes_k \mathcal{O}_{D_J^\circ}, \tag{5.3}$$

where

$$D_J^\circ = D_J \setminus \cup_{i\notin J} D_i.$$

Using the element τ we identify the right side of (5.3) with the trivial line bundle and obtain an isomorphism

$$\nu_J : \mathcal{L}_J^{\otimes d_J} \xrightarrow{\sim} \mathcal{O}_{D_J^\circ},$$

where

$$\mathcal{L}_J := \bigotimes_{j\in J} \left(\mathcal{N}_{j|_{D_J^\circ}}\right)^{\otimes d_j/d_J}.$$

Define $\bar{D}_J^\circ$ as the locus of points in the total space of the line bundle $\mathcal{L}_J$, whose d_J-th tensor power is mapped to 1 by ν_J. By construction, $\bar{D}_J^\circ \to D_J^\circ$ is a $\boldsymbol{\mu}_{d_J}$-torsor.

Step 4. Define

$$\rho_\tau^{\boldsymbol{\mu}}([X/K]) := \sum_{\emptyset\neq J\subseteq I} (-1)^{\#J-1}[\bar{D}_J^\circ \times \mathbb{A}^{\#J-1} \to D_J^\circ \times \mathbb{A}^{\#J-1}, d_J] \tag{5.4}$$

Theorem 18. *The additive homomorphism*

$$\rho_\tau^{\boldsymbol{\mu}} : \mathrm{Burn}(K) \to \mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k)$$

is well-defined and is a ring homomorphism.

Proof. The verification that $\rho_\tau^{\boldsymbol{\mu}}$ is well-defined is completely analogous to the proof of Theorem 4, we only need to keep track of the torsor structures, which is straightforward because all intersections of components of snc divisors on the blowup are birationally equivalent to projective bundles.

The main difficulty is the proof of multiplicativity. The issue we face is that the fiber product of two snc models over $\mathrm{Spec}(\mathfrak{o})$ is not, in general, an snc model: it has toric singularities of a particular type. To resolve this issue is suffices to provide an explicit resolution of singularities of affine toric varieties of the following type:

$$\prod_{j\in J} x_j^{a_j} = \prod_{j'\in J'} y_{j'}^{a'_{j'}},$$

where J, J' are nonempty, and $a_j, a'_{j'} \geq 1$, for all $j \in J, j' \in J'$. Moreover, the family of resolutions should be compatible with restrictions, when some of the variables are set to 1. Using the toric language, we translate this problem to a combinatorial construction: for every pair $r, r' \in \mathbb{Z}_{\geq 1}$ and vectors

$$\mathbf{a} := (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 1}^r, \quad \mathbf{a}' := (a'_1, \ldots, a'_{r'}) \in \mathbb{Z}_{\geq 1}^{r'}$$

consider the cone

$$\Lambda(\mathbf{a}, \mathbf{a}') := \{(\mathbf{u}, \mathbf{u}') \in \mathbb{R}_{\geq 0}^{r+r'} \mid \langle \mathbf{u}, \mathbf{a} \rangle = \langle \mathbf{u}', \mathbf{a}' \rangle\} \subset \mathbb{R}^{r+r'},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product.

Note that all boundary strata of $\Lambda(\mathbf{a}, \mathbf{a}')$ are naturally isomorphic to $\Lambda(\mathbf{b}, \mathbf{b}')$, for some integral vectors $\mathbf{b}, \mathbf{b}'$ of smaller dimension (up to permutation of indices).

Then choose fans $\Sigma(\mathbf{a}, \mathbf{a}')$ supported in $\Lambda(\mathbf{a}, \mathbf{a}')$, for all $\mathbf{a}, \mathbf{a}'$, such that the collection of these fans satisfies the following properties:

- All faces $\sigma \in \Sigma(\mathbf{a}, \mathbf{a}')$ are primitive and simplicial, i.e., the corresponding toric variety $X_{\Sigma(\mathbf{a}, \mathbf{a}')}$ is smooth.
- The choice of fans is covariant with respect to symmetric groups $\mathfrak{S}_r \times \mathfrak{S}_{r'}$ acting on the indices.
- For all $\mathbf{a}, \mathbf{a}'$, the induced fan on a boundary of $\Lambda(\mathbf{a}, \mathbf{a}')$ coincides with the corresponding fan $\Sigma(\mathbf{b}, \mathbf{b}')$.

The existence of such an equivariant, compatible with restriction to boundary, family of fans can be proved inductively (see, e.g.,[CTHS05]).

Now consider two snc models $\mathcal{X}, \mathcal{X}'$ of smooth proper K-varieties X and X', with function fields $K(X), K(X')$. A choice of a family of fans as above determines an snc model $\mathcal{W}$ over $\mathrm{Spec}(\mathfrak{o})$ of the product $W := X \times X'$, endowed with a natural map

$$\mathcal{W} \to \mathcal{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathcal{X}'.$$

The special fiber $\mathcal{W}_0$ is stratified, with strata labelled by:

(i) nonempty subsets $J \subseteq I$, $J' \subseteq I'$,
(ii) $\sigma \in \Sigma(\mathbf{a}, \mathbf{a}')$ such that its interior is contained in the interior of $\Lambda(\mathbf{a}, \mathbf{a}')$, where the entries of the vector $\mathbf{a}$, respectively, $\mathbf{a}'$, are d_j, $j \in J$, respectively, $d_{j'}$, $j' \in J'$, (with some enumeration of elements in J, J').

For each triple (J, J', σ) as above, the corresponding open stratum $D^\circ_{J,J',\sigma}$ is a disjoint union of open snc strata in $\mathcal{W}_0$. Moreover, it is the base of a $\boldsymbol{\mu}_{\mathrm{lcm}(d_J, d_{J'})}$-torsor

$$\bar{D}^\circ_{J,J',\sigma} \to D^\circ_{J,J',\sigma}.$$

Lemma 19. *There exists a natural isomorphism of* $\boldsymbol{\mu}_{\mathrm{lcm}(d_J,d_{J'})}$*-torsors*

$$\begin{array}{ccc}\bar{D}^\circ_{J,J',\sigma} & \xrightarrow{\sim} & \bar{D}^\circ_J \times \bar{D}^\circ_{J'} \times \mathbb{T}_\sigma \\ \downarrow & & \downarrow \\ D^\circ_{J,J',\sigma} & \xrightarrow{\sim} & (D^\circ_J \times D^\circ_{J'})/\boldsymbol{\mu}_{\mathrm{lcm}(d_J,d_{J'})} \times \mathbb{T}_\sigma\end{array}$$

where $\mathbb{T}_\sigma$ is a split torus over k.

Proof. This is a straightforward verification. ☐

Fix nonempty J, J'. We substitute (3.2) into

$$\rho_{\dim(X)}([K(X)/K]) \cdot \rho_{\dim(X')}([K(X')/K])$$

and extract products of terms corresponding to J and J'. We claim that the sum of these terms equals to the sum over σ of contributions of snc strata in $\mathcal{W}_0$, labelled by J, J' and σ as in (ii) above. This claim follows from the equality (Euler characteristic computation):

$$\sum_\sigma (-1)^{\dim(\sigma)} = (-1)^{\dim(\Lambda(\mathbf{a},\mathbf{a}'))},$$

where σ are subject to (ii). Multiplicativity of ρ now follows by taking summation over all J, J'. ☐

Remark 20. The dependence of the homomorphism $\rho^{\boldsymbol{\mu}}_\tau$ on the choice of the cotangent vector τ is quite simple. Namely, define a group homomorphism

$$\gamma : k^\times \to \mathrm{Aut}(\mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k))$$

as follows: Assume that we are given an element $t \in k^\times$ and an additive generator $[\bar{X} \to X, d]$ of $\mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k)$, where $d \in \mathbb{Z}_{\geq 1}$, and $\bar{X} \to X$ is a $\boldsymbol{\mu}_d$-torsor over a smooth scheme X. Define

$$\gamma(t)([\bar{X} \to X, d])$$

as the class $[\bar{X}' \to X, d]$, where $\bar{X}' \to X$ is the twist of $\boldsymbol{\mu}_d$-torsor $\bar{X} \to X$ by the canonical $\boldsymbol{\mu}_d$-torsor

$$\mathrm{Spec}(k[x]/(x^d - t)) \to \mathrm{Spec}(k)\,.$$

Then for any $\tau, \tau' \in (\mathfrak{m}/\mathfrak{m}^2) \setminus \{0\}$ such that $\tau' = t \cdot \tau, t \in k^\times$ one has

$$\rho^{\boldsymbol{\mu}}_{\tau'} = \gamma(t) \circ \rho^{\boldsymbol{\mu}}_\tau\,.$$

We now explain the relation between the multiplicative specialization homomorphism we constructed and $\hat{\rho}$ defined in (5.2). Consider additive maps

$$\mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k) \xrightarrow{\psi} \mathrm{Burn}(k)[T] \xrightarrow{T=1} \mathrm{Burn}(k)$$

given by

$$[Y' \to Y, d] \mapsto [Y/k]T^d \mapsto [Y/k].$$

We have

$$\hat{\rho} = \psi \circ \rho_\tau^{\boldsymbol{\mu}},$$

and specializing $T = 1$ we obtain ρ from (3.1). The advantage of the nonmultiplicative homomorphisms ρ and $\hat{\rho}$ is that they do not depend on the choice of the cotangent vector τ.

There is also a *ring homomorphism*

$$\chi : \mathrm{Burn}^{\hat{\boldsymbol{\mu}}}(k) \to \mathrm{Burn}(k)$$

given by

$$[Y' \to Y, d] \mapsto [Y'/k].$$

Composing $\rho_\tau^{\boldsymbol{\mu}}$ with this homomorphism we obtain a ring homomorphism

$$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

depending on τ (as in Remark 20). This is the Burnside ring version of motivic reduction/motivic volume [NP17, Theorem 2.6.1], [HK06].

Remark 21. These finer homomorphisms can be applied to considerations in Section 4, leading to more refined invariants.

References

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572, 2002.

[Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.

[Bor14] Lev Borisov. Class of the affine line is a zero divisor in the Grothendieck ring, 2014. `arXiv:1412.6194`.

[CTHS05] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov. Compactification équivariante d'un tore (d'après Brylinski et Künnemann). *Expo. Math.*, 23(2):161–170, 2005.

[CTP16] Jean-Louis Colliot-Thélène and Alena Pirutka. Hypersurfaces quartiques de dimension 3: non-rationalité stable. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(2):371–397, 2016.

[dFF13] Tommaso de Fernex and Davide Fusi. Rationality in families of threefolds. *Rend. Circ. Mat. Palermo (2)*, 62(1):127–135, 2013.

[HK06] Ehud Hrushovski and David Kazhdan. Integration in valued fields. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 261–405. Birkhäuser Boston, Boston, MA, 2006.

[HKT16] Brendan Hassett, Andrew Kresch, and Yuri Tschinkel. Stable rationality and conic bundles. *Math. Ann.*, 365(3-4):1201–1217, 2016.

[LL03] Michael Larsen and Valery A. Lunts. Motivic measures and stable birational geometry. *Mosc. Math. J.*, 3(1):85–95, 259, 2003.

[NP17] Johannes Nicaise and Sam Payne. A tropical motivic Fubini theorem with applications to Donaldson-Thomas theory, 2017. `arXiv:1703.10228`.

[NS17] Johannes Nicaise and Evgeny Shinder. The motivic nearby fiber and degeneration of stable rationality, 2017. `arXiv:1708.027901`.

[Tim82] Klaus Timmerscheidt. On deformations of three-dimensional rational manifolds. *Math. Ann.*, 258(3):267–275, 1981/82.

[Tot16] Burt Totaro. Hypersurfaces that are not stably rational. *J. Amer. Math. Soc.*, 29(3):883–891, 2016.

[Voi15] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

[Wło03] Jarosław Włodarczyk. Toroidal varieties and the weak factorization theorem. *Invent. Math.*, 154(2):223–331, 2003.

Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: `maxim@ihes.fr`

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA