INTERSECTIONS OF THREE QUADRICS IN $\mathbb{P}^7$

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Abstract. We study rationality properties of smooth complete intersections of three quadrics in $\mathbb{P}^7$. We exhibit a smooth family of such intersections with both rational and non-rational fibers.

1. Introduction

The specialization method, introduced by Voisin [Voi15], and developed by Colliot-Thélène–Pirutka, Totaro, and others, has led to major advances in higher-dimensional complex birational geometry. It makes it possible, for the first time, to prove failure of stable rationality of some smooth quartic threefolds [CTP16b], cyclic covers [Voi15], [Bea16], [CTP16a], [Oka16], and large degree smooth Fano hypersurfaces in projective space [Tot16].

The specialization method yields failure of stable rationality of a very general member of a family of complex algebraic varieties from the existence of a single, mildly singular, fiber with an explicit obstruction, that can be formulated in terms of integral decomposition of the diagonal or universal CH_0-triviality (see Section 2.1 for more details and references). A surprising aspect of applications of the method was that *a priori* different families of varieties admit specializations to the same 'reference varieties'. This allows us to propagate the failure of stable rationality, by finding suitable chains of specializations. Examples of such 'reference varieties' are conic or quadric surface bundles over rational surfaces, with carefully chosen discriminant loci (see [Pir17]). A similar approach – via specialization to quartic del Pezzo fibrations over $\mathbb{P}^1$ – may be used to essentially settle the stable rationality problem for very general smooth rationally connected threefolds [HKT16], [HT16], [KO17], with the exception of cubic threefolds, whose stable rationality remains elusive [Voi17].

New effects arise in dimension four: rationality properties can change in smooth families [HPT16a]. The relevant reference variety is $Y \subset$

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$\mathbb{P}^2_\lambda \times \mathbb{P}^3_y$, given by the vanishing of the $(2,2)$ form

$$\lambda_1\lambda_2 y_0^2 + \lambda_0\lambda_2 y_1^2 + \lambda_0\lambda_1 y_2^2 + F(\lambda_0,\lambda_1,\lambda_2)y_3^2, \tag{1.1}$$

with

$$F(\lambda_0,\lambda_1,\lambda_2) := \lambda_0^2+\lambda_1^2+\lambda_2^2 - 2(\lambda_0\lambda_1+\lambda_0\lambda_2+\lambda_1\lambda_2) \tag{1.2}$$

defining a conic tangent to each coordinate line. The family is the universal $(2,2)$ hypersurface, a Fano fourfold of Picard rank two.

The variety Y gives rise to other interesting families of fourfolds failing stable rationality: double covers [HPT16b], and conic bundles over $\mathbb{P}^3$ [APBvB16]. In this note, we exhibit another natural family of smooth complex projective fourfolds X with rational and irrational fibers: Fano fourfolds of Picard rank one, obtained as intersections of three quadrics in $\mathbb{P}^7$.

Theorem 1. *Let $B \subset \mathrm{Gr}(3, \Gamma(\mathcal{O}_{\mathbb{P}^7}(2)))$ be the open subset of the Hilbert scheme parametrizing smooth complete intersections of three quadrics in $\mathbb{P}^7$ and*

$$\phi : \mathcal{X} \to B \tag{1.3}$$

the corresponding universal family.

1. *For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational.*
2. *The set of $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B for the Euclidean topology.*

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2. Strategy

We follow the approach in [HPT16a]. In this section, we recall the main steps in the proof; details are provided in Section 3.

2.1. Fibers that are not stably rational.

Recall that a projective variety X over a field k is *universally* CH_0-*trivial* if for all field extensions k'/k the natural degree homomorphism from the Chow group of zero-cycles

$$\mathrm{CH}_0(X_{k'}) \to \mathbb{Z}$$

is an isomorphism. A projective morphism

$$\beta : \tilde{X} \to X$$

of k-varieties is *universally* CH_0*-trivial* if for all extensions k'/k the push-forward homomorphism

$$\beta_* : \mathrm{CH}_0(\tilde{X}_{k'}) \to \mathrm{CH}_0(X_{k'})$$

is an isomorphism.

In this paper, we apply the specialization method of Voisin in the following form.

Theorem 2. [Voi15, Theorem 2.1], [CTP16b, Theorem 2.3] *Let*

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ *such that the fiber*

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(R) X *admits a desingularization*

$$\beta : \tilde{X} \to X$$

such that the morphism β *is universally* CH_0*-trivial;*

(O) *the variety* $\tilde{X}$ *is not universally* CH_0*-trivial.*

Then a very general fiber of ϕ *is not universally* CH_0*-trivial; in particular, it is not stably rational.*

Condition (O) holds, for instance, if the unramified cohomology group $H^2_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/2)$ is nontrivial. By [CTP16b, Proposition 1.8] and [CTP16a, Lemma 2.4] condition (R) is satisfied if for every scheme point x of X, the fiber $\beta^{-1}(x)$, considered as a variety over the residue field $\kappa(x)$, could be written as $\beta^{-1}(x) = \cup_i X_i$, where each component X_i is smooth, geometrically irreducible and $\kappa(x)$-rational and each intersection $X_i \cap X_j$ is either empty or has a zero-cycle of degree 1.

In [HPT16a, Propositions 11, 12], we constructed a hypersurface $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2, 2)$, satisfying the obstruction condition (O) and the resolution condition (R) as above (see (1.1)). The first projection $Y \to \mathbb{P}^2$ endows Y with a structure of a quadric surface bundle with discriminant curve of degree 8. As explained in [Bea77, Exemple 1.4.4], smooth intersections of three quadrics in $\mathbb{P}^7$ are also birational to quadric surface bundles over $\mathbb{P}^2$, with discriminant curve of degree 8 (see Proposition 6 below). These two families, hypersurfaces of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ and intersections of three quadrics in $\mathbb{P}^7$, are genuinely different; see Section 4 for a precise statement. Both specialize (birationally) to the same reference fourfold: in Proposition 7 we provide an explicit example of a (singular) intersection of three quadrics $X \subset \mathbb{P}^7$ such that X is birational to the variety Y above. We deduce Theorem 1, Part (1), from Theorem 2 at the end of Section 3.1.

2.2. **One rational fiber.**

Let $\phi : \mathcal{X} \to B$ be the family (1.3). By Proposition 6, for any $b \in B$, the fiber $\mathcal{X}_b$ is birational to a quadric bundle over $\mathbb{P}^2$. In Section 3.2 (Proposition 9), we provide an explicit example of a fiber $\mathcal{X}_b$, birational to a quadric bundle with a rational section. In particular, the fourfold $\mathcal{X}_b$ is rational.

2.3. **Density of rational fibers.**

Let $X \subset \mathbb{P}^7$ be a smooth intersection of three quadrics. As in the previous step, in order to establish that X is rational, it suffices to exhibit a quadric surface bundle $\pi : Q \to \mathbb{P}^2$ such that Q is birational to X and such that π admits a rational section. By Springer's theorem, it suffices to show that π has a rational multisection of odd degree. For quadric bundles this can be formulated as a Hodge-theoretic condition:

Proposition 3. [CTV12, Corollaire 8.2] *Let Q be a smooth projective complex algebraic variety, admitting a dominant morphism $\pi : Q \to \mathbb{P}^2$, with generic fiber a quadric of dimension at least* 1. *Then the integral Hodge conjecture holds for classes of degree* $(2,2)$ *on* Q.

Thus, in order to show that X is rational, it suffices to provide a $(2,2)$-Hodge class intersecting the class of a fiber of π in odd degree. We achieve this by studying the infinitesimal period map. This technique is explained in [Voi07, 5.3.4].

The Hodge diamond of X is of the following form:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 3 & & 38 & & 3 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

In particular, the degree 4 cohomology is essentially of weight 2. We can then apply the following criterion to the family $\mathcal{X} \to B$ of Theorem 1 (cf. [Voi07, 5.3.4]):

Proposition 4. *Suppose there exists a $b_0 \in B$ and $\gamma \in H^{2,2}(\mathcal{X}_{b_0})$ such that the infinitesimal period map*

$$\bar{\nabla} : T_{B,b_0} \to Hom(H^{2,2}(\mathcal{X}_{b_0}), H^{1,3}(\mathcal{X}_{b_0})), \tag{2.1}$$

evaluated at γ, gives a surjective map

$$\bar{\nabla}(\gamma) : T_{B,b_0} \to H^{1,3}(\mathcal{X}_{b_0}). \tag{2.2}$$

Then for any $b \in B$ and any Euclidean neighborhood $b \in B' \subset B$, the image of the natural map (composition of inclusion with local trivialization):

$$\mathcal{H}^{2,2}_{\mathbb{R}} \to H^4(\mathcal{X}_b, \mathbb{R}) \tag{2.3}$$

contains an open subset $V_b \subset H^4(\mathcal{X}_b, \mathbb{R})$. Here $\mathcal{H}^{2,2}_{\mathbb{R}}$ is a vector bundle over B' with fiber over u equal to the real classes of type $(2, 2)$ in $H^4(\mathcal{X}_u)$.

In order to check the infinitesimal criterion we use an explicit description of the period map:

Proposition 5. [Ter90, Corollary 2.5, Proposition 2.6] *Let $X \subset \mathbb{P}^7$ be a smooth complete intersection of three quadrics, defined by equations*

$$Q_i(x_0, \dots, x_7) = 0, \quad i = 0, 1, 2$$

and let

$$F = \mu_0 Q_0 + \mu_1 Q_1 + \mu_2 Q_2 \in \mathbb{C}[\mu_0, \mu_1, \mu_2, x_0, \dots, x_7].$$

Let $I \subset \mathbb{C}[\mu_0, \mu_1, \mu_2, x_0, \dots, x_7]$ be the ideal generated by

$$\partial F/\partial \mu_i, \quad i = 0, 1, 2 \text{ and } \partial F/\partial x_i, \quad i = 0, \dots, 7.$$

Put

$$R = \mathbb{C}[\mu_0, \mu_1, \mu_2, x_0, \dots, x_7]/I$$

and let $R_{(a,b)}$ be the space of homogeneous elements of degree (a, b) in R, with respect to the grading (μ, x). Then there is an isomorphism

$$H^{4-q,q}_{\text{prim}}(X) \simeq R_{(q,2q-2)}$$

and the period map (2.1) is identified with the multiplication homomorphism

$$R_{(1,2)} \otimes R_{(2,2)} \to R_{(3,4)}. \tag{2.4}$$

Recall that the primitive cohomology $H^{p,q}_{\mathrm{prim}}$ is the cokernel of the natural map $H^{p,q}(\mathbb{P}^7) \to H^{p,q}(X)$.

In Section 3.3, we provide an explicit example $X = \mathcal{X}_{b_0}$ such that the period map 2.4 is surjective (Proposition 10). Theorem 1, Part (2), then follows. In fact, by Proposition 3.2, there exists a smooth intersection of three quadrics birational to a quadric bundle with a rational section. Similarly to [HPT16a, Proposition 14] the density of rational fibers follows from the infinitesimal criterion that we verify in Proposition 10.

3. Computations

We work over the complex numbers. We first recall the construction of Beauville [Bea77, Exemple 1.4.4]:

Proposition 6. *Let $X \subset \mathbb{P}^7$ be a smooth complete intersection of three quadrics. Then X is birational to a quadric bundle over $\mathbb{P}^2$, with discriminant curve of degree 8.*

Concretely, let $\ell \subset X$ be a line and $G_\ell \simeq \mathbb{P}^5$ the space of 2-planes $\Pi \subset \mathbb{P}^7$ containing ℓ. Then X is birational to a quadric surface bundle

$$\pi : Q \to \mathbb{P}^2,$$

where $Q \subset \mathbb{P}^2 \times G_\ell$ is given by

$$Q = \{([\lambda_0 : \lambda_1 : \lambda_2], \Pi) | \quad \{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0\} \supset \Pi\}. \tag{3.1}$$

More explicitly, assume that the line is given by equations

$$\ell : x_2 = x_3 = \ldots = x_7 = 0$$

and write, for $i = 0, 1, 2$,

$$Q_i = x_0 L_i(x_2, x_3, \ldots, x_7) + x_1 M_i(x_2, x_3, \ldots, x_7) + q_i(x_2, x_3, \ldots, x_7),$$

where L_i and M_i are linear forms and q_i is quadratic. Any 2-plane $\Pi \subset \mathbb{P}^7$ containing ℓ intersects the 5-plane $x_0 = x_1 = 0$ in a unique point $[0 : 0 : x_2 : \cdots : x_7]$. This allows us to identify the space of 2-planes $\Pi \subset \mathbb{P}^7$ containing ℓ with $\mathbb{P}^5$. Then the quadric bundle (3.1) is defined in $\mathbb{P}^2 \times \mathbb{P}^5$ by the equations

$$\sum_{i=0}^{2} \lambda_i L_i(x_2, x_3, \ldots, x_7) = \sum_{i=0}^{2} \lambda_i M_i(x_2, x_3, \ldots, x_7) = \\ = \sum_{i=0}^{2} \lambda_i q_i(x_2, x_3, \ldots, x_7) = 0. \tag{3.2}$$

3.1. **Fibers that are not stably rational.** Let $X \subset \mathbb{P}^7$ be the intersection of three quadrics

$$(3.3)\quad Q_0:\ -x_0x_5 + x_3^2 + x_4x_6 - 2x_5^2 = 0; \quad Q_1:\ x_0x_5 + x_1x_4 + x_2^2 - 2x_5^2 = 0; \quad Q_2: x_0x_7 - x_1x_6 + x_5^2 + x_7^2 = 0.$$

Note that X contains a line $\ell : x_2 = \ldots = x_7 = 0$. Using equations (3.2), we obtain that X is birational to a quadric bundle $Q \to \mathbb{P}^2$, defined in $\mathbb{P}^2 \times \mathbb{P}^5$ as an intersection of two forms of bidegree $(1,1)$ and one form of bidegree $(1,2)$:

$$(3.4)\quad (\lambda_0 - \lambda_1)x_5 = \lambda_2 x_7, \qquad \lambda_1 x_4 = \lambda_2 x_6 \quad \lambda_1 x_2^2 + \lambda_0 x_3^2 + \lambda_0 x_4 x_6 + (\lambda_2 - 2\lambda_0 - 2\lambda_1)x_5^2 + \lambda_2 x_7^2 = 0.$$

In the open set $\lambda_2 \neq 0$ we can define X by a single equation

$$\lambda_1 x_2^2 + \lambda_0 x_3^2 + \frac{\lambda_0\lambda_1}{\lambda_2} x_4^2 + (\frac{(\lambda_0 - \lambda_1)^2}{\lambda_2} + \lambda_2 - 2\lambda_0 - 2\lambda_1)x_5^2 = 0,$$

hence, X is birational to a hypersurface $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,2)$ defined by

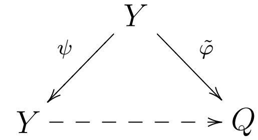
$$(3.5)\qquad \lambda_1\lambda_2 x_2^2 + \lambda_0\lambda_2 x_3^2 + \lambda_0\lambda_1 x_4^2 + F(\lambda_0,\lambda_1,\lambda_2)x_5^2 = 0,$$

where $F(\lambda_0,\lambda_1,\lambda_2) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda_0\lambda_1 - 2\lambda_0\lambda_2 - 2\lambda_1\lambda_2$.
This is precisely the hypersurface we considered in [HPT16a, Propositions 11, 12].

Proposition 7. *Let $Q \subset \mathbb{P}^2 \times \mathbb{P}^5$ be defined by the equations (3.4) and let $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ be the hypersurface given by the equation* (3.5). *Then the birational map*

$$(3.6)\quad \varphi : Y \dashrightarrow Q, \quad (\lambda_0 : \lambda_1 : \lambda_2, x_2 : \ldots : x_5) \mapsto (\lambda_0 : \lambda_1 : \lambda_2, \lambda_2 x_2 : \lambda_2 x_3 : \lambda_2 x_4 : \lambda_2 x_5 : \lambda_1 x_4 : (\lambda_0 - \lambda_1)x_5)$$

extends to the following diagram



where the morphisms $\psi : \tilde{Y} \to Y$ *and* $\tilde{\varphi} : \tilde{Y} \to Q$ *are birational and universally* CH_0*-trivial.*

Proof. First note that φ is indeed a birational map between Y and Q. The locus $Y^{nd} \subset Y$ where the map φ is not defined is a union of three components
$Y_1 : \lambda_2 = 0, x_4 = x_5 = 0$;
$Y_2 : \lambda_1 = \lambda_2 = 0, x_5 = 0$;
$Y_3 : \lambda_0 - \lambda_1 = 0, \lambda_2 = 0, x_4 = 0$.
Note that Y_1 is isomorphic to a product $\mathbb{P}^1_{\lambda_0:\lambda_1} \times \mathbb{P}^1_{x_2:x_3}$, and similarly Y_2 is isomorphic to a projective plane $\mathbb{P}^2_{x_2:x_3:x_4}$ with homogeneous coordinates $[x_2 : x_3 : x_4]$ and $Y_3 \simeq \mathbb{P}^2_{x_2:x_3:x_5}$.

We construct $\tilde{Y}$ by successive blowups of Y_1, the proper transform of Y_2 and the proper transform of Y_3. After each blowup we verify:

- the indeterminacy locus of φ on the blowup;
- the universal CH_0-triviality of fibers of the extension of φ to the blowup and of the blowup map. In each case we obtain that the corresponding fiber is either reduced to a point or projective (or affine, if we compute on open charts) spaces. We provide details for the first computations and the expressions in the coordinates for the remaining charts.

Blowup of Y_1. We have three charts:

(1) $U_1 : x_4 = \lambda_2 u_4,\ x_5 = \lambda_2 u_5$, the exceptional divisor is given by $\lambda_2 = 0$. Since we blow up the locus $\lambda_2 = 0, x_4 = x_5 = 0$, we consider one of the charts $\lambda_0 \neq 0$ or $\lambda_1 \neq 0$ of $\mathbb{P}^2$ and one of the charts $x_2 \neq 0$ or $x_3 \neq 0$ of $\mathbb{P}^3$.
We extend φ to a birational map $\varphi_1 : U_1 \dashrightarrow Q$,

$$(\lambda_0, \lambda_1, \lambda_2, x_2, x_3, u_4, u_5) \mapsto (\lambda_0, \lambda_1, \lambda_2, x_2, x_3, \lambda_2 u_4, \lambda_2 u_5, \lambda_1 u_4, (\lambda_0 - \lambda_1) u_5).$$

Since one of coordinates λ_0, λ_1 is nonzero, and one of coordinates x_2, x_3 is nonzero, we have that φ_1 is well-defined. The image of φ_1 is contained in the closure of the image of φ, hence it is contained in Q, so that we obtain a map $\varphi_1 : U_1 \to Q$.

The image of the exceptional divisor is the set of points

$$E_1 = (\lambda_0, \lambda_1, 0, x_2, x_3, 0, 0, \lambda_1 u_4, (\lambda_0 - \lambda_1) u_5).$$

Then for any field $k'/\mathbb{C}$ and for any point $P \in E_1(k')$ the fiber $\varphi_1^{-1}(P)$ is either a point or a line (if $\lambda_1 = 0$ or $\lambda_0 - \lambda_1 = 0$), which ensures the universal CH_0-triviality of the map φ_1 on this chart.

The equation defining U_1 is

$$\lambda_1 x_2^2 + \lambda_0 x_3^2 + \lambda_0\lambda_1\lambda_2 u_4^2 + F(\lambda_0, \lambda_1, \lambda_2)\lambda_2 u_5^2 = 0.$$

Let $\psi_1 : U_1 \to Y$ be the blowup map. Then, the image I_1 of the exceptional divisor is given by the conditions

$$\lambda_2 = 0, \quad \lambda_1 x_2^2 + \lambda_0 x_3^2 = 0.$$

The latter condition defines a point since the coordinates $\lambda_0 : \lambda_1$ and $x_2 : x_3$ are homogeneous. Then for any field $k'/\mathbb{C}$ and for any point $P \in I_1(k')$ the fiber $\psi_1^{-1}(P)$ is a plane with coordinates u_4 and u_5, which ensures the universal CH_0-triviality of the map ψ_1 on this chart.

(2) U_2 :

- change of variables:

$$\lambda_2 = x_4\lambda_2', x_5 = x_4 u_5;$$

- equation defining the blowup:

$$\lambda_1\lambda_2' x_2^2 + \lambda_0\lambda_2^2 x_3^2 + \lambda_0\lambda_1 x_4 + F(\lambda_0, \lambda_1, \lambda_2' x_4)x_4 u_5^2 = 0.$$

- exceptional divisor:

$$x_4 = 0, \lambda_1\lambda_2' x_2^2 + \lambda_0\lambda_2^2 x_3^2 = 0.$$

- extension of φ is given by:

$$(\lambda_0, \lambda_1, \lambda_2' x_4, \lambda_2' x_2, \lambda_2' x_3, \lambda_2' x_4, \lambda_2' x_4 u_5, \lambda_1, (\lambda_0 - \lambda_1)u_5).$$

- domain, where the extension is not defined is the proper transform Y_2' of Y_2:

$$\lambda_1 = \lambda_2' = 0, u_5 = 0.$$

- the image of the exceptional divisor:

$$(\lambda_0, \lambda_1, 0, \lambda_2' x_2, \lambda_2' x_3, 0, 0, \lambda_1, (\lambda_0 - \lambda_1)u_5).$$

(3) U_3 :

- change of variables:

$$\lambda_2 = x_5\lambda_2', x_4 = x_5 u_4;$$

- equation defining the blowup:

$$\lambda_1\lambda_2' x_2^2 + \lambda_0\lambda_2' x_3^2 + \lambda_0\lambda_1 x_5 u_4^2 + F(\lambda_0, \lambda_1, \lambda_2' x_5)x_5 = 0;$$

- exceptional divisor:

$$x_5 = 0, \lambda_1\lambda_2'x_2^2 + \lambda_0\lambda_2^2x_3^2 = 0;$$

- extension of φ is given by:

$$(\lambda_0, \lambda_1, \lambda_2'x_5, \lambda_2'x_2, \lambda_2'x_3, \lambda_2'x_5u_4, \lambda_2'x_5, \lambda_1u_4, \lambda_0 - \lambda_1)$$

- domain, where the extension is not defined is the proper transform Y_3' of Y_3:

$$\lambda_0 - \lambda_1 = \lambda_2' = 0, u_4 = 0.$$

- the image of the exceptional divisor:

$$(\lambda_0, \lambda_1, 0, \lambda_2'x_2, \lambda_2'x_3, 0, 0, \lambda_1u_4, \lambda_0 - \lambda_1).$$

Blowup of the proper transforms Y_2' and Y_3'

Note that Y_2 and Y_3, and hence their proper transforms, do not intersect. Hence we can use charts U_2 and U_3 independently for their blowups.

(1) On the chart U_2:

(a)
- change of variables:

$$\lambda_1 = \lambda_2'\lambda_1', u_5 = \lambda_2'v_5$$

- exceptional divisor:

$$\lambda_2' = 0, \lambda_0x_3^2 + \lambda_0\lambda_1'x_4;$$

- extension of φ is everywhere defined:

$$(\lambda_0, \lambda_1'\lambda_2', x_4\lambda_2', x_2, x_3, x_4, \lambda_2'x_4v_5, \lambda_1', (\lambda_0 - \lambda_1'\lambda_2')v_5);$$

- the image of the exceptional divisor:

$$(1, 0, 0, x_2, x_3, x_4, x_4v_5, \lambda_1', v_5).$$

(b)
- change of variables:

$$\lambda_2' = \lambda_1\lambda_2'', u_5 = \lambda_1v_5;$$

- exceptional divisor:

$$\lambda_1 = 0, \lambda_0\lambda_2''x_3^2 + \lambda_0x_4 = 0;$$

- extension of φ is everywhere defined:

$$(\lambda_0 : \lambda_1 : \lambda_1\lambda_2'', \lambda_2''x_2, \lambda_2''x_3, \lambda_2''x_4, \lambda_1\lambda_2''x_4v_5, 1, v_5(\lambda_0 - \lambda_1));$$

- the image of the exceptional divisor:

$$(1, 0, 0, \lambda_2''x_2, \lambda_2''x_3, \lambda_2''x_4, 0, 1, v_5).$$

(c)
- change of variables:
$$\lambda_2' = u_5\lambda_2'', \lambda_1 = u_5\lambda_1'';$$
- exceptional divisor:
$$u_5 = 0, \lambda_0\lambda_2''x_3^2 + \lambda_0\lambda_1''x_4 = 0;$$
- extension of φ is everywhere defined:
$$(\lambda_0, \lambda_1''u_5, \lambda_2''u_5, \lambda_2''x_2, \lambda_2''x_3, \lambda_2''x_4, \lambda_2''x_4u_5, \lambda_1'', \lambda_0 - \lambda_1''u_5);$$
- the image of the exceptional divisor:
$$(1, 0, 0, \lambda_2''x_2, \lambda_2''x_3, \lambda_2''x_4, 0, \lambda_1'', 1).$$

(2) On the chart U_3:

(a)
- change of variables:
$$\lambda_0 - \lambda_1 = \lambda_2'\lambda_0', u_4 = \lambda_2'v_4;$$
- exceptional divisor:
$$\lambda_2' = 0, \lambda_1x_2^2 + \lambda_1x_3^2 - 4\lambda_1x_5 = 0;$$
- extension of φ is everywhere defined:
$$(\lambda_1 + \lambda_2'\lambda_0', \lambda_1, \lambda_2'x_5, x_2, x_3, \lambda_2'x_5v_4, x_5, \lambda_1v_4, \lambda_0');$$
- the image of the exceptional divisor:
$$(\lambda_1, \lambda_1, 0, x_2, x_3, 0, x_5, \lambda_1v_4, \lambda_0').$$

(b)
- change of variables:
$$\lambda_2' = (\lambda_0 - \lambda_1)\lambda_2'', u_4 = (\lambda_0 - \lambda_1)v_4;$$
- exceptional divisor:
$$(\lambda_0 - \lambda_1) = 0, \lambda_1\lambda_2''x_2^2 + \lambda_1\lambda_2''x_3^2 - 4\lambda_1\lambda_2''x_5 = 0;$$
- extension of φ is everywhere defined:
$$(\lambda_0, \lambda_1, \lambda_2''(\lambda_0 - \lambda_1)x_5, \lambda_2''x_2, \lambda_2''x_3, (\lambda_0 - \lambda_1)\lambda_2''x_5v_4, \lambda_2''x_5, \lambda_1v_4, 1);$$
- the image of the exceptional divisor:
$$(\lambda_1, \lambda_1, 0, \lambda_2''x_2, \lambda_2''x_3, 0, \lambda_2''x_5, \lambda_1v_4, 1).$$

(c)
- change of variables:
$$\lambda_2' = u_4\lambda_2'', \lambda_0 - \lambda_1 = u_4\lambda_0';$$
- exceptional divisor:
$$u_4 = 0, \lambda_1\lambda_2''x_2^2 + \lambda_1\lambda_2''x_3^2 - 4\lambda_1\lambda_2''x_5 = 0;$$

- extension of φ is everywhere defined:

$$(\lambda_1 + u_4\lambda_0', \lambda_1, \lambda_2'' u_4 x_5, \lambda_2'' x_2, \lambda_2'' x_3, \lambda_2'' x_5 u_4, \lambda_2'' x_5, \lambda_1, \lambda_0');$$

- the image of the exceptional divisor:

$$(\lambda_1, \lambda_1, 0, \lambda_2'' x_2, \lambda_2'' x_3, 0, \lambda_2'' x_5, \lambda_1, \lambda_0').$$

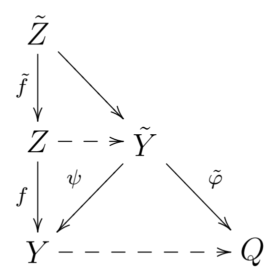
□

Corollary 8. *Let $Q \subset \mathbb{P}^2 \times \mathbb{P}^5$ be defined by the equations (3.4). Then Q admits a resolution of singularities $\beta : \tilde{Q} \to Q$ such that*

(i) *the variety $\tilde{Q}$ is not universally* CH_0*-trivial;*
(ii) *the map β is a universally* CH_0*-trivial morphism.*

Proof. We use Proposition 7: Q is birational to a variety Y with $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) \neq 0$ by [HPT16a, Proposition 11]. In particular, property (i) holds for any resolution $\tilde{Q}$ of Q.

In [HPT16a, Proposition 12] we constructed a resolution of singularities $f : Z \to Y$ such that f is a universally CH_0-trivial morphism. Then there is birational map $\tilde{f} : \tilde{Z} \to Z$ with $\tilde{Z}$ smooth, such that the rational map $Z \dashrightarrow \tilde{Y}$ extends to a map $\tilde{Z} \to \tilde{Y}$:



Note that the map $\tilde{f}$ is universally CH_0-trivial: by weak factorization, $\tilde{f}$ factors through blow-ups and blow-downs at smooth centers, each of these maps is universally CH_0-trivial. Hence, in the diagram above, the maps $\tilde{f}, f, \psi, \tilde{\varphi}$ are universally CH_0-trivial. We deduce from the diagram that the composite map $\tilde{Z} \to Q$ is also universally CH_0-trivial, which shows (ii). □

Proof of Theorem 1, Part (1):
From Theorem 2 and Corollary 8 we deduce that a very general quadric bundle defined by equations (3.2) is not universally CH_0-trivial. In particular, there exists a smooth intersection of three quadrics X birational to a smooth quadric bundle Q defined by an equation of type (3.2), such that Q is not universally CH_0-trivial. Since universal CH_0-triviality is a birational invariant of smooth projective varieties, we deduce that

X is not universally CH_0-trivial. Then Theorem 1, Part (1), follows directly from Theorem 2, applied to the universal family $\phi : \mathcal{X} \to B$ of smooth complete intersections of three quadrics in $\mathbb{P}^7$. $\square$

3.2. **One rational fiber.** Consider the quadrics

$$\begin{aligned} Q_0 : \quad & x_0(x_3 + x_5 + 2x_6 + 3x_7) + x_1(-x_5 + 5x_6 + 2x_7) - \\ & - x_2x_3 - x_2x_4 + x_2x_5 + x_3^2 - x_4x_6 + x_5^2 + x_6^2 + x_7^2 = 0; \\ Q_1 : \quad & x_0(-x_2 + 3x_5 + 7x_6 + 11x_7) + x_1(x_4 + 9x_5 + 4x_6 + x_7) + \\ & + x_2^2 - x_2x_3 + 2x_3x_6 + x_4^2 + 3x_4x_7 + 2x_5^2 + 3x_6^2 + 5x_7^2 = 0; \\ Q_2 : \quad & x_0(11x_5 + 13x_6 + 8x_7) + x_1(-x_3 + 6x_5 + 7x_6 + 3x_7) + \\ & + x_2^2 + 5x_2x_7 - x_3x_4 + 9x_3x_5 + 13x_5^2 + 4x_6^2 + 11x_7^2 = 0. \end{aligned}$$

Proposition 9. *Let $X \subset \mathbb{P}^7$ be the intersection*

$$Q_0 = Q_1 = Q_2 = 0$$

Then X is smooth and rational.

Proof. A Magma [BCP97] computation shows that X is smooth. Furthermore, X contains a line

$$\ell : x_2 = \ldots = x_7 = 0.$$

As in Proposition 6, considering the space $G_\ell \simeq \mathbb{P}^5$ of 2-planes $\Pi \subset \mathbb{P}^7$ containing ℓ, we find that X is birational to a fibration in quadrics $Q \to \mathbb{P}^2$, where $Q \subset \mathbb{P}^2 \times G_\ell$,

$$Q = \{([\lambda_0 : \lambda_1 : \lambda_2], \Pi) | \quad \{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0\} \supset \Pi\}.$$

The first projection $Q \to \mathbb{P}^2$ admits a rational section: the plane containing ℓ and the point $[0 : 0 : \lambda_0 : \lambda_1 : \lambda_2 : 0 : 0 : 0]$ is contained in $\{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0\}$. Indeed, by (3.2), we have that $Q \subset \mathbb{P}^2 \times \mathbb{P}^5$ is defined by the equations:

$$\lambda_0(x_3+x_5+2x_6+3x_7)+\lambda_1(-x_2+3x_5+7x_6+11x_7)+\lambda_2(11x_5+13x_6+8x_7) = 0$$
$$\lambda_0(-x_5+5x_6+2x_7)+\lambda_1(x_4+9x_5+4x_6+x_7)+\lambda_2(-x_3+6x_5+7x_6+3x_7) = 0$$
$$\lambda_0(-x_2x_3-x_2x_4+x_2x_5+x_3^2-x_4x_6+x_5^2+x_6^2+x_7^2)+\lambda_1(x_2^2-x_2x_3+2x_3x_6+x_4^2+$$
$$+3x_4x_7+2x_5^2+3x_6^2+5x_7^2)+\lambda_2(x_2^2+5x_2x_7-x_3x_4+9x_3x_5+13x_5^2+4x_6^2+11x_7^2) = 0$$

and, substituting

$$[x_2 : x_3 : \ldots : x_7] = [0 : 0 : \lambda_0 : \lambda_1 : \lambda_2 : 0 : 0 : 0],$$

we obtain

$$\lambda_0\lambda_1 - \lambda_0\lambda_1 = 0, \quad \lambda_1\lambda_2 - \lambda_1\lambda_2 = 0,$$
$$\lambda_0(-\lambda_0\lambda_1 - \lambda_0\lambda_2 + \lambda_1^2) + \lambda_1(\lambda_0^2 + \lambda_2^2 - \lambda_0\lambda_1) + \lambda_2(\lambda_0^2 - \lambda_1\lambda_2) = 0.$$

□

3.3. **Density of rational fibers.** Using the notation of Section 3.2, consider quadrics

$$\begin{array}{rcl} Q_0' & := & Q_0 + x_0^2 + x_5^2 \\ Q_1' & := & Q_1 \\ Q_2' & := & Q_2 + x_1^2 + x_3^2 \end{array}$$

Proposition 10. *Let $X' \subset \mathbb{P}^7$ be the intersection*

$$Q_0' = Q_1' = Q_2' = 0.$$

Then X' is smooth and there exists a $\gamma \in H^{2,2}(X')$ such that the period map (2.2) is surjective.

Proof. A Magma computation shows that X' is smooth. In order to compute the period map we use expression (2.4). We used Macaulay2 [GS] to verify that the following monomials

$$\{\mu_0\mu_2^2x_7^4,\ \mu_1\mu_2^2x_7^4,\ \mu_2^3x_7^4\}$$

form a basis of the graded part $R_{(3,4)} \simeq H^{1,3}(X')$. In particular $\gamma = \mu_2^2x_7^2$ works. □

4. Differentiating quadric bundles

The goal of this section is to show that the quadric bundles arising from complete intersection of three quadrics in $\mathbb{P}^7$ do in fact differ from the $(2,2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ considered in [HPT16a]. Note however that both families specialize to the *same* reference variety (1.1).

Let $\pi : Q \to \mathbb{P}^2$ be a quadric surface bundle with smooth degeneracy curve $D \subset \mathbb{P}^2$, i.e., Q is a smooth complex projective fourfold, π is a flat morphism with smooth ($\simeq \mathbb{F}_0$) fibers over $\mathbb{P}^2 \setminus D$, and quadric cones ($\simeq \mathbb{P}(1,1,2)$) as fibers over D. Let $\tau : S \to \mathbb{P}^2$ denote the associated double cover, simply branched along D. We may interpret S as the Stein factorization of the relative variety of lines

$$F_1(Q/\mathbb{P}^2) \to S \to \mathbb{P}^2;$$

as such, S is equipped with a natural conic bundle structure and thus a class $\alpha_Q \in H^2(S, \mu_2)$. We refer the reader to [APS15] for a close analysis of the equivalence between quadric surface bundles and Azumaya algebras over double covers.

We present a cohomological interpretation of this correspondence due to Laszlo [Las89]. Let $H^2_0(S, \mathbb{Z})$ denote the primitive cohomology of S, i.e., the kernel of τ_*. It carries the structure of a lattice with respect to the intersection form, as well as a weight two Hodge structure. Choose an embedding

$$\begin{array}{ccc} Q & \hookrightarrow & \mathbb{P}(E) \\ & \overset{\pi}{\searrow} & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

where $E \to \mathbb{P}^2$ is a rank four vector bundle. Let $H^4_0(Q, \mathbb{Z})$ denote kernel of the push forward homomorphism

$$H^4(Q, \mathbb{Z}) \to H^6(\mathbb{P}(E), \mathbb{Z}).$$

This carries the structure of a lattice and a weight four Hodge structure. Let $H^4_0(Q, \mathbb{Z})(1)$ denote its Tate twist, a weight two Hodge structure; this reverses the sign of the integral quadratic form.

Theorem 11. [Las89, Th. II.3.1] *There exists an embedding of abelian groups*

$$\Phi : H^4_0(Q, \mathbb{Z})(1) \hookrightarrow H^2_0(S, \mathbb{Z})$$

compatible with the lattice and Hodge structures. The image has index two and is characterized as follows:

$$\mathrm{image}(\Phi) = \Lambda_Q := \{\gamma \in H^2_0(S, \mathbb{Z}) : (\gamma \bmod 2, \alpha_Q) \equiv 0 \bmod 2\}.$$

Now suppose we have a birational equivalence

$$\begin{array}{ccccc} Q_1 & & \overset{\sim}{\dashrightarrow} & & Q_2 \\ & \searrow & & \swarrow & \\ & & \mathbb{P}^2 & & \end{array}$$

of quadric bundles over $\mathbb{P}^2$. It is clear that Q_1 and Q_2 must have the same degeneracy curve $D \subset \mathbb{P}^2$ and induced double cover $\tau : S \to \mathbb{P}^2$. Consider the classes $\alpha_{Q_1}, \alpha_{Q_2} \in \mathrm{Br}(S)[2]$, obtained via the canonical surjection $H^2(S, \mu_2) \to \mathrm{Br}(S)[2]$. Since α_{Q_i} generates the kernel of

$$H^2(\mathbb{C}(S), \mu_2) \to H^2(\mathbb{C}(Q_i), \mu_2)$$

by [Ara75, p.469], we have $\alpha_{Q_1} = \alpha_{Q_2}$.

Proposition 12. *Let $D \subset \mathbb{P}^2$ be a very general octic plane curve, $Q_1, Q_2 \to \mathbb{P}^2$ quadric surface bundles with degeneracy curve D, where $Q_1 \subset \mathbb{P}^2 \times \mathbb{P}^3$ is a $(2,2)$ hypersurface and $Q_2 \subset \mathbb{P}^2 \times \mathbb{P}^5$ is a complete intersection of hypersurfaces of bidegrees $(1,1)$, $(1,1)$, $(1,2)$. Then Q_1 and Q_2 are not birational over $\mathbb{P}^2$.*

The precise condition we require is that $\mathrm{Pic}(S) \simeq \mathbb{Z}$.

Proof. For the first example, let h_1 and h_2 denote the pull-backs of the hyperplane classes from each factor. Then we have $[Q_1] = 2h_1 + 2h_2$ and

	h_1^2	h_1h_2	h_2^2
h_1^2	0	0	2
h_1h_2	0	2	2
h_2^2	2	2	0

For the second example, let g_1 and g_2 denote the hyperplace classes as above so that

$$[Q_2] = 4g_1^2g_2 + 5g_1g_2^2 + 2g_2^3.$$

Then we have

	g_1^2	g_1g_2	g_2^2
g_1^2	0	0	2
g_1g_2	0	2	5
g_2^2	2	5	4

These two lattices are inequivalent over the 2-adics. Indeed, their ranks modulo two differ. It follows that the lattices $H^4_0(Q_1, \mathbb{Z})$ and $H^4_0(Q_2, \mathbb{Z})$ are also inequivalent, as a nondegenerate lattice and its orthogonal complement in a unimodular lattice have the same discriminant groups up to sign. (The discriminant groups are a way of packaging the p-adic invariants of a lattice.)

Under our assumption, $\mathrm{Br}(S)[2] = H^2(S, \mu_2)/\langle h\rangle$ where h is the hyperplane class pulled back from $\mathbb{P}^2$. If Q_1 and Q_2 were birational over $\mathbb{P}^2$ then

$$\alpha_{Q_1} = \alpha_{Q_2} \in H^2(S, \mu_2)/\langle h\rangle,$$

whence $\Lambda_{Q_1} \simeq \Lambda_{Q_2}$. This would contradict Theorem 11. □

Remark 13. Observe that the common reference variety (1.1) admits nontrivial 2-torsion in its unramified cohomology. It is intriguing that we differentiate the smooth members through a 2-adic computation of lattices.

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