HOMOMORPHISMS OF MULTIPLICATIVE GROUPS OF FIELDS PRESERVING ALGEBRAIC DEPENDENCE

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Abstract. We study homomorphisms of multiplicative groups of fields preserving algebraic dependence and show that such homomorphisms give rise to valuations.

Introduction

In this paper we formulate and prove a version of the Grothendieck section conjecture. For function fields of algebraic varieties over algebraically closed ground fields, this conjecture states, roughly, that the existence of group-theoretic sections of homomorphisms of their absolute Galois groups implies existence of geometric sections of morphisms of models of these fields.

In detail, let k be an algebraically closed field, X an irreducible algebraic variety over k, and $K = k(X)$ its function field. Let G_K be the absolute Galois group of K. Fix a prime ℓ not equal to the characteristic of k and let $\mathcal{G}_K$ be the maximal pro-ℓ-quotient of G_K, the Galois group of the maximal ℓ-extension of K. Write

$$\mathcal{G}_K^a = \mathcal{G}_K/[\mathcal{G}_K, \mathcal{G}_K] \quad \text{and} \quad \mathcal{G}_K^c := \mathcal{G}_K/[\mathcal{G}_K, [\mathcal{G}_K, \mathcal{G}_K]],$$

for the abelianization and its canonical central extension:

$$1 \to \mathcal{Z}_K \to \mathcal{G}_K^c \xrightarrow{\pi_a} \mathcal{G}_K^a \to 1. \tag{1}$$

Let $\Sigma_K = \Sigma(\mathcal{G}_K^c)$ be the set of topologically noncyclic subgroups $\sigma \subset \mathcal{G}_K^a$ whose preimages $\pi_a^{-1}(\sigma) \subset \mathcal{G}_K^c$ are abelian. It is known that function fields $K = k(X)$ of transcendence degree ≥ 2 over $k = \bar{\mathbb{F}}_p$ are determined, modulo purely inseparable extensions, by the pair $(\mathcal{G}_K^a, \Sigma_K)$ [5], [7], and [14].

This raises the question of *functoriality*, i.e., the reconstruction of rational morphisms between algebraic varieties from continuous homomorphisms of absolute Galois groups of their function fields. This general fundamental question was proposed by Grothendieck and lies at the core of the Anabelian Geometry Program.

Key words and phrases. Field theory, valuations.

The main open problem in this program relates to a Galois-theoretic criterium for the existence of rational sections of fibrations. Let

$$\pi : X \to Y,$$

be a fibration of integral algebraic varieties over k with geometrically irreducible generic fiber of dimension at least 1 over a base Y of dimension ≥ 2. This defines a field embedding

$$\pi^* : k(Y) \hookrightarrow k(X),$$

with the image of $L := k(Y)$ algebraically closed in $K := k(X)$. Dually, we have a surjective homomorphism of absolute Galois groups (a restriction map)

$$G_K \to G_L,$$

as well as the induced homomorphisms

$$\mathcal{G}^c_K \to \mathcal{G}^c_L, \quad \mathcal{G}^a_K \to \mathcal{G}^a_L.$$

A minimalistic version of Grothendieck's *Section conjecture*, over algebraically closed k, would be:

Conjecture 1. *Assume that $\pi_a : \mathcal{G}^a_K \to \mathcal{G}^a_L$ admits a section*

$$\xi_a : \mathcal{G}^a_L \to \mathcal{G}^a_K \tag{2}$$

such that

$$\xi_a(\Sigma_L) \subset \Sigma_K. \tag{3}$$

Then there exist a finite purely inseparable extension

$$\iota^* : L \hookrightarrow L' = k(Y')$$

and a rational map

$$\xi : Y' \to X,$$

such that

$$\xi^* \circ \pi^*(L) = \iota^*(L) \subset L'.$$

Thus $\xi(Y')$ is a section over Y, modulo purely inseparable extensions.

Conjecture 1 is closely related to questions considered in this note. Recall that, by Kummer theory,

$$\mathcal{G}^a_K = \mathrm{Hom}(K^\times, \mathbb{Z}_\ell(1)),$$

and that (2) induces the dual homomorphism of pro-ℓ-completions of the multiplicative groups

$$\hat{\psi} : \hat{K}^\times \to \hat{L}^\times.$$

Then (3) says that $\hat{\psi}$ respects the skew-symmetric pairings on $\hat{K}^\times$ and $\hat{L}^\times$, with values in the second Galois cohomology group of the corresponding field (with ℓ-torsion coefficients). The groups $\hat{K}^\times$ and $\hat{L}^\times$ contain $K^\times/k^\times$ and $L^\times/k^\times$, respectively. If the restriction ψ of $\hat{\psi}$ to $K^\times/k^\times$ satisfies

$$\psi : K^\times/k^\times \subseteq L^\times/k^\times \subset \hat{L}^\times,$$

then ψ respects algebraic dependence, mapping algebraically dependent elements in $K^\times$ to algebraically dependent elements of $L^\times$ (modulo $k^\times$). For function fields this is equivalent to (3) (see, e.g., [6, Section 5]). This relates the "minimalistic" version of the Section conjecture for "rational" maps to our main result, which we now explain.

From now on, let K be an arbitrary field over k. Let ν be a nonarchimedean valuation of K, i.e., a homomorphism

$$\nu : K^\times \to \Gamma_\nu$$

onto a totally ordered group such that the induced map

$$\nu : K \to \Gamma_\nu \cup \{\infty\}, \quad \nu(0) = \infty,$$

satisfies a nonarchimedean triangle inequality. Let

$$\mathfrak{m}_{K,\nu} \subset \mathfrak{o}_{K,\nu}, \quad \boldsymbol{K}_\nu := \mathfrak{o}_{K,\nu}/\mathfrak{m}_{K,\nu}, \quad \mathfrak{o}_{K,\nu}^\times,$$

be the maximal ideal, valuation ring, residue field, and units with respect to ν, respectively. If $K|k$ is a (transcendental) field extension and ν a valuation of K, then its restriction to k is also a valuation; and we have

$$\mathfrak{o}_{K,\nu}^\times \cap k^\times = \mathfrak{o}_{k,\nu}^\times, \quad \mathfrak{o}_{K,\nu}^\times/\mathfrak{o}_{k,\nu}^\times \subseteq K^\times/k^\times,$$

and a natural surjection

$$\mathfrak{o}_{K,\nu}^\times/\mathfrak{o}_{k,\nu}^\times \longrightarrow\!\!\!\!\rightarrow \boldsymbol{K}_\nu^\times/\boldsymbol{k}_\nu^\times.$$

We consider extensions of fields

$$k \subseteq \tilde{k} \subseteq \tilde{k}_a \subset K,$$

where k is the prime subfield of K, i.e., $k = \mathbb{F}_p$ or $\mathbb{Q}$, and $\tilde{k}_a \subset K$ the algebraic closure of $\tilde{k}$ in K, i.e., the set of all algebraic elements over $\tilde{k}$ contained in K. Assume that $\bar{x}_1, \bar{x}_2 \in K^\times/k^\times$ satisfy

$$\operatorname{tr}\deg_{\tilde{k}}(\tilde{k}(x_1, x_2)) \leq 1, \tag{4}$$

for their lifts $x_1, x_2 \in K^\times$; and this does not depend on the choice of lifts. We write $x_1 \sim_{\tilde{k}} x_2$ and say that x_1 and x_2 are contained in the same one-dimensional field; clearly $1 \sim_{\tilde{k}} x$, for all $\bar{x} \in K^\times/k^\times$. From now on, we use the same notation for an element $x \in K^\times$ and its image in $K^\times/k^\times$. Let

$$l \subseteq \tilde{l} \subseteq \tilde{l}_a \subset L$$

be field extensions, where l is the prime subfield of L, $\tilde{l}_a$ the algebraic closure of $\tilde{l}$ in L, and let

$$\psi : K^\times/k^\times \to L^\times/\tilde{l}^\times$$

be a homomorphism of multiplicative groups. We say that ψ preserves algebraic dependence with respect to $\tilde{k}, \tilde{l}$ if

$$x_1 \sim_{\tilde{k}} x_2 \quad \Rightarrow \quad \psi(x_1) \sim_{\tilde{l}} \psi(x_2), \quad \forall\, x_1, x_2 \in K^\times/k^\times.$$

Theorem 2. *Let $k \subseteq \tilde{k} \subset K$ and $l \subseteq \tilde{l} \subset L$ be field extensions as above. Assume that $\tilde{l} = \tilde{l}_a$ and that there exists a homomorphism*

$$\psi : K^\times/k^\times \to L^\times/\tilde{l}^\times, \tag{5}$$

such that

- *ψ preserves algebraic dependence with respect to $\tilde{k}$ and $\tilde{l}$;*
- *there exist*
$$y_1, y_2 \in \psi(K^\times/k^\times), \qquad \text{such that} \qquad y_1 \not\sim_{\tilde{l}} y_2;$$
- *ψ satisfies Assumption (AD) of Section 7.*

Then either

(P) *there exists a field $F \subset K$ such that ψ factors through*
$$K^\times/k^\times \twoheadrightarrow K^\times/F^\times,$$

(V) *there exists a nontrivial valuation ν on K such that the restriction of ψ to*
$$\mathfrak{o}^\times_{K,\nu}/\mathfrak{o}^\times_{k,\nu} \subseteq K^\times/k^\times$$
is trivial on
$$(1+\mathfrak{m}_\nu)^\times/\mathfrak{o}^\times_{k,\nu}$$
and it factors through the reduction map
$$\mathfrak{o}^\times_{K,\nu}/\mathfrak{o}^\times_{k,\nu} \twoheadrightarrow \boldsymbol{K}^\times_\nu/\boldsymbol{k}^\times_\nu \to L^\times/\tilde{l}^\times,$$

(VP) *there exist a nontrivial valuation ν on K and a field $\boldsymbol{F}_\nu \subset \boldsymbol{K}_\nu$ such that the restriction of ψ to $\mathfrak{o}^\times_{K,\nu}/\mathfrak{o}^\times_{k,\nu}$ factors through*
$$\mathfrak{o}^\times_{K,\nu}/\mathfrak{o}^\times_{k,\nu} \twoheadrightarrow \boldsymbol{K}^\times_\nu/\boldsymbol{F}^\times_\nu \to L^\times/\tilde{l}^\times.$$

Note that we do not assume that k is algebraically closed. In the geometric setting treated in [8], when $K = \tilde{k}(X)$ is a function field of an algebraic variety X over $\tilde{k} = \bar{\mathbb{F}}_p$, case (P) corresponds to projections, the center of the valuation ν arising in case (V) is, birationally, the image of the section, and the above theorem can be viewed as a "rational" version of the minimalistic section conjecture (case (VP) corresponds to valuations composed with projections).

To see this connection in more detail, recall that $K^\times$ admits a natural homomorphism with dense image to $\hat{K}^\times$, the dual to $\mathcal{G}^a_K$. The usual form of the section conjecture, as in Conjecture 1, is equivalent to the statement about homomorphisms

$$\psi : \hat{K}^\times \to \hat{L}^\times$$

such that pairs $\hat{f}, \hat{g}$ with $(\hat{f}, \hat{g}) = 0$ map to a pairs with $(\psi(\hat{f}), \psi(\hat{g})) = 0$. Note that the image of $K^\times \subset \hat{K}^\times$ plays the role of a $\mathbb{Z}$-sublattice, in the geometric case of function fields of algebraic varieties over algebraically closed fields. The corresponding statement of the theorem in the case of a function field is indeed a rational version of the section conjecture; we expect that it can be deduced from our version, using the fact that the natural sublattice $K^\times \subset \hat{K}^\times$ is necessarily mapped into $L^\times \subset \hat{L}^\times$, modulo multiplication by a constant $a \in \mathbb{Z}_\ell^\times$ (as in [5]).

Note that the abelian-by-central version of section conjecture does not hold for big fields. Indeed, as it was pointed in [3], the maximal extensions coprime to ℓ of function fields have isomorphic Galois group, depending only on the algebraically closed ground field and the dimension. However, our theorem says that we can still obtain valuations from the multiplicative group homomorphisms respecting algebraic dependence.

Here we extend the argument in [8] from function fields to arbitrary fields, under the additional technical assumption (AD) on ψ, which holds for K of positive characteristic. Although we believe that the main theorem holds in full generality, i.e., without the (AD) assumption, we were forced to add it, due to purely technical difficulties in our treatment of valuations of K which extend p-valuations of $\mathbb{Q}$. To achieve clarity of the presentation, we decided to remove such valuations from present considerations.

Related results on connections between Galois groups, valuations, and projective geometry can be found in [1], [9], [10], [11], [13].

The idea of the proof is to reduce the problem to a question in plane projective geometry over the prime subfield k, as in [4] and [5]. We view $\mathbb{P}(K) := K^\times/k^\times$ as a projective space over k. To establish Theorem 2, it suffices to show the existence of a subgroup $\mathfrak{U} \subset K^\times/k^\times$ such that:

Condition 3. For every projective line $\mathfrak{l} \subset \mathbb{P}(K)$, $\mathfrak{U} \cap \mathfrak{l}$ is either

(1) the line $\mathfrak{l}$,
(2) a point $\mathfrak{q} \in \mathfrak{l}$,
(3) the affine line $\mathfrak{l} \setminus \mathfrak{q}$, or
(4) if $k = \mathbb{Q}$, a set projectively equivalent to

$$\mathbb{Z}_{(p)} \subset \mathbb{A}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{Q}),$$

the set of rational numbers with the denominator coprime to p.

Indeed, such a subgroup is necessarily either $F^\times/k^\times$ for some subfield $F \subset K$, or $\mathfrak{o}_{K,\nu}$, for some valuation ν (see Section 7). By construction, the homomorphism ψ will satisfy the cases (P) or (V) in Theorem 2, respectively.

To find such $\mathfrak{U}$, we use the results of [12] and [2]. First we deduce that the restriction of ψ to every plane $\mathbb{P}^2 \subset \mathbb{P}(K)$ is either an embedding or is induced by a natural construction from some nonarchimedean valuation (see Section 5). We distinguish two cases:

- there exists a line $\mathfrak{l} \subset \mathbb{P}(K)$ such that the restriction of ψ to $\mathfrak{l}$ is injective,
- no such lines exist.

In the first case, property (4) of Condition 3 does not occur, and the proof works uniformly for $k = \mathbb{F}_p$ or $\mathbb{Q}$. In this case, we construct $\mathfrak{U}$ by first taking the union $\mathfrak{u}$ of all lines $\mathfrak{l}(1, x)$ on which ψ is injective and then putting $\mathfrak{U} := \mathfrak{u} \cdot \mathfrak{u}$. We should that $\mathfrak{U}$ is closed under multiplication and its intersections with projective subspaces $\Pi \subset \mathbb{P}_k(K)$ define a flag structure on Π, and thus a valuation on K.

In the second case, the proofs are slightly different, leading to a case-by-case analysis in Section 5.

Acknowledgments. We are very grateful to the referees for their careful reading of the first version of the paper, which helped us to improve the exposition. This work was partially supported by Laboratory of Mirror Symmetry NRU HSE, RF grant 14.641.31.0001. The first and second authors were funded by the Russian Academic Excellence Project '5-100'. The first author was also supported by a Simons Fellowship and by EPSRC program grant EP/M024830. The third author was partially supported by NSF grant 1601912.

1. Projective geometry

Let $\mathbb{P}$ be a projective space over a field k, i.e., the projectivization of a vector space over k. Let $\Pi(\mathfrak{q}_0, \dots, \mathfrak{q}_n) \subseteq \mathbb{P}$ the projective envelope of points $\mathfrak{q}_0, \dots, \mathfrak{q}_n \in \mathbb{P}$. Working with lines and planes, we write

$$\mathfrak{l} = \mathfrak{l}(\mathfrak{q}_0, \mathfrak{q}_1), \quad \text{resp.} \quad \Pi = \Pi(\mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{a}_2),$$

for a projective line through $\mathfrak{q}_0, \mathfrak{q}_1$, or a plane through $\mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2$.

Let ν a nonarchimedean valuation of k, $\mathfrak{o} = \mathfrak{o}_\nu$ the corresponding valuation ring, and $\boldsymbol{k}_\nu$ the residue field. Fixing a lattice

$$\Lambda \simeq \mathfrak{o}^{n+1} \hookrightarrow k^{n+1},$$

we obtain a natural surjection

$$\rho = \rho_\Lambda : \mathbb{P}^n(k) \to \mathbb{P}^n(\boldsymbol{k}_\nu). \tag{6}$$

A *3-coloring* of $\mathbb{P}^2(k)$ is a surjection

$$\mathfrak{c} : \mathbb{P}^2(k) \to \{\bullet, \circ, \star\}, \tag{7}$$

onto a set of 3 elements, such that

- every $\mathfrak{l} \subset \mathbb{P}^2(k)$ is colored in exactly two colors, i.e., $\mathfrak{c}(\mathfrak{l})$ consists of two elements.

A 3-coloring is called *trivial of type*

- *I*: if there exists a line $\mathfrak{l} \subset \mathbb{P}^2$ such that $\mathfrak{c}$ is constant on $\mathbb{P}^2 \setminus \mathfrak{l}$,
- *II*: if there exists a point $\mathfrak{q} \in \mathbb{P}^2(k)$ such that for every $\mathfrak{l} \subset \mathbb{P}^2$ containing $\mathfrak{q}$, $\mathfrak{c}$ is constant on $\mathfrak{l} \setminus \mathfrak{q}$.

It was discovered in the early 1980s that such colorings are related to valuations, see, e.g., [12]. The same structure resurfaced in the study of the *commuting* elements of Galois groups of function fields in [4], exhibiting unexpected projective structures *within* $\mathcal{G}^a_K$. This was a crucial step in the recognition of inertia and decomposition subgroups in $\mathcal{G}^a_K$.

Precisely, we have (see [12, Theorem 2] and [4]):

Proposition 4. *Assume that $\mathbb{P}^2(k)$ carries a 3-coloring. Then there exists a nonarchimedean valuation ν such that the coloring $\mathfrak{c}$ in (7) is induced from a trivial covering*

$$\mathfrak{c}_\nu : \mathbb{P}^2(\boldsymbol{k}_\nu) \to \{\bullet, \circ, \star\},$$

for some ρ as in (6).

2. Flag maps

We will consider maps (respectively, homomorphisms)

$$f : \mathbb{P} \to A$$

from projective spaces over k to a set (respectively, an abelian group). The map f is called a *flag map* if its restriction f_Π to every finite dimensional projective subspace $\Pi \subset \mathbb{P}$ is a flag map. For $k = \mathbb{F}_p$ and

$$f : \mathbb{P}^n(\mathbb{F}_p) \to A,$$

this means that there exists a flag of projective subspaces

$$\mathbb{P}^n \supset \mathbb{P}^{n-1} \ldots \supset \mathbb{P}^1 \supset \mathbb{P}^0 = \mathfrak{q} \tag{8}$$

such that f is constant on $\mathbb{P}^i(\mathbb{F}_p) \setminus \mathbb{P}^{i-1}(\mathbb{F}_p)$, for all $i = 1, \ldots, n$. For $k = \mathbb{Q}$ and

$$f : \mathbb{P}^n(\mathbb{Q}) \to A,$$

this means that either

- there is a flag as in (8) so that f is constant on $\mathbb{P}^i(\mathbb{Q}) \setminus \mathbb{P}^{i-1}(\mathbb{Q})$, for all $i = 1, \ldots, n$, or
- there exist a prime p, a surjection
$$\rho = \rho_\Lambda : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{P}^n(\mathbb{F}_p)$$
as in (6), and a flag map
$$\bar{f} : \mathbb{P}^n(\mathbb{F}_p) \to A,$$
such that
$$f = \bar{f} \circ \rho.$$

Proposition 5. [4, Theorem 6.3.4] *Let*
$$f : \mathbb{P}(K) = K^\times / k^\times \to A$$
be a group homomorphism which is also a flag map. Then there exist a valuation ν *of* K *and a homomorphism* $r : \Gamma_\nu \to A$ *such that* f *factors through*
$$K^\times / k^\times \xrightarrow{\nu} \Gamma_\nu \xrightarrow{r} A.$$

A map f on $\mathbb{P}^n(k)$ that is a flag map on every hyperplane defines an

$$\begin{array}{rcl} \hat{f} : \hat{\mathbb{P}}^n & \to & A \\ \lambda & \mapsto & f_{\mathrm{gen}}(\lambda) \end{array} \qquad (9)$$

on the dual space, by assigning to a projective hyperplane the *generic* value of f on this hyperplane, i.e., the constant value on the complement to a codimension one subspace of that hyperplane.

Every map
$$f : \mathbb{P}^2(\mathbb{F}_2) \to \{0, 1\}$$
has the property that its restriction $f_{\mathfrak{l}}$ to any line $\mathfrak{l} \subset \mathbb{P}^2(\mathbb{F}_2)$ is a flag map, but not all such f are flag maps. The following theorem, generalizing results in [4, Section 2], shows that this is the only exception.

Theorem 6. *Let* $f : \mathbb{P}^n(k) \to A$ *be a map such that* $f_{\mathfrak{l}}$ *is a flag map, for every* $\mathfrak{l} \subset \mathbb{P}^n(k)$. *Then* f *is a flag map, unless* $k = \mathbb{F}_2$ *and* $\mathbb{P}^n$ *contains a plane* $\Pi = \mathbb{P}^2$ *such that* f *is not a flag map on* Π.

Proof. We proceed by induction, assuming that f is a flag map on every $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, $n \geq 2$. We separate the cases:

- $\hat{f}$ is constant,
- $\hat{f}$ takes at least two values.

In the first case, let a be the generic value of f on hyperplanes and $S \subset \mathbb{P}^n$ be such that $f(x) \neq a$, for $x \in S$. The projective span $\mathbb{P}(S)$ of S is a proper subspace of $\mathbb{P}^n$, of codimension at least 2. Indeed, consider a subset of distinct elements $x_i \in S, i = 1, \dots, n$; it contains an element with generic value of f on the span $\mathbb{P}(x_1, \dots, x_n)$. Hence, by assumption, $\dim \mathbb{P}(x_1, \dots, x_n) \leq n-2$. Thus the dimension of $\mathbb{P}(S)$ is also bounded by $(n-2)$. In particular, $f(x) = a$, for all $x \in \mathbb{P}^n \setminus \mathbb{P}(S)$. By induction, f is flag on $\mathbb{P}(S)$, thus also on $\mathbb{P}^n$.

In the second case, let $a_\lambda := f_{\mathrm{gen}}(\lambda)$ be the generic value of f on $\Pi_\lambda \subset \mathbb{P}^n, \lambda \in \hat{\mathbb{P}}^n$. We have two possibilities:

(1) There is a λ_1, with $f_{\mathrm{gen}}(\lambda_1) = a_1$, so that for $\mathfrak{q} \in \mathbb{P}^n \setminus \Pi_{\lambda_1}$, one has $f(\mathfrak{q}) \neq a_1$.
(2) There are $\lambda_1 \neq \lambda_2$, with different generic values a_1, a_2, such that there are points $\mathfrak{q}_i \in \mathbb{P}^n \setminus \Pi_{\lambda_i}$ with $f(\mathfrak{q}_i) = a_i$.

In Case (1), f is constant outside of Π_{λ_1} and hence a flag map, by induction. Indeed, let $\mathbb{P}^r \subset \Pi_{\lambda_1}$ be a projective subspace with $f = a_1$ on $\Pi_{\lambda_1} \setminus \mathbb{P}^r$. On any line $\mathfrak{l}(x_1, x_2) \subset \mathbb{P}^n$, with $x_1, x_2 \notin \Pi_{\lambda_1}$, intersecting $\Pi_{\lambda_1} \setminus \mathbb{P}^r$ at some point z_1, f is constant on $\mathfrak{l}(x_1, x_2) \setminus z_1$. Hence $f(x_1) = f(x_2)$ in this case. It remains to show that for $x_i \in \mathbb{P}^n \setminus \Pi_{\lambda_1}$ with $z_1 \in \mathbb{P}^r$ we also have $f(x_1) = f(x_2)$. Consider the $\Pi(x, z_1, \mathfrak{q})$, for any $\mathfrak{q} \in \Pi_{\lambda_1} \setminus \mathbb{P}^{r_i}$. If $z_2 \in \mathfrak{l}(z_1, \mathfrak{q}), z_2 \neq \mathfrak{q}, z_1$, then $\mathfrak{l}(x_, z_1)$ and $\mathfrak{l}(x_1, z_2)$ intersect at some point $w \in \mathbb{P}^n \setminus \Pi_{\lambda_1}$ and hence $f(x_1) = f(w) = f(x_2)$, which concludes the proof.

Case (2) does not occur unless $n = 2$. Indeed, let $\Pi_{x_1, \mathfrak{q}_1}$ be a hyperplane containing $\mathfrak{q}_1$ and intersecting $\Pi_{\lambda_1} \setminus \mathbb{P}^r$ nontrivially. The latter contains an affine plane $\mathbb{A}^{n-2}_x$ in the intersection $\mathbb{P}^{n-2}_{1,x} = \mathbb{P}^{n-1}_{x, \mathfrak{q}_1} \cap \mathbb{P}^{n-1}_1$ and $\mathbb{A}^{n-2}_x$ spans $\mathbb{P}^{n-2}_{1,x}$. Thus $\mathbb{P}^{n-1}_x$ is spanned by $\mathfrak{q}_1$ and $\mathbb{A}^{n-2}_x$ and hence a_1 is the generic value on $\mathbb{P}^{n-1}_{x, \mathfrak{q}_1}$. These hyperplanes are parametrized by $\hat{\mathbb{P}}^{n-1}_1 \subset \hat{\mathbb{P}}^n$ minus a subset of hyperplanes which do not intersect $\Pi_{\lambda_1} \setminus \mathbb{P}^r$. This set is empty if $r < n-2$ and consists of one point $\mathfrak{p}_1$ if $r = n-2$. Applying the same argument to a_2 we obtain a different $\hat{\mathbb{P}}^{n-1}_2 \subset \hat{\mathbb{P}}^n$. The hyperplanes $\hat{\mathbb{P}}^{n-1}_1, \hat{\mathbb{P}}^{n-1}_1$ intersect at $\hat{\mathbb{P}}^{n-2}$ and we obtain a contradiction if $n - 2 > 0$, since $\hat{\mathbb{P}}^{n-2}$ contains at least 3-points. Thus we obtain hyperplanes in $\mathbb{P}^n$ with two generic values, contradicting the inductive assumption.

We have reduced to $n = 2$, with the additional assumption that f is nonconstant on any line.

Lemma 7. *Case (2) does not occur for $n = 2$ unless $k = \mathbb{F}_2$.*

Proof. Consider Π_{λ_1} and its subset of generic points, which contains $\mathbb{A}^1_{\lambda_1}$. Any line from $\mathfrak{q}_1$ to a point in $\mathbb{A}^1_{\lambda_1}$ has generic value a_1. Let E_1

be the union of all such lines and $E(a_1) \subseteq E_1$ the subset of points with value a_1. We define E_2 and $E_2(a_2)$ in a similar fashion. If $k = \mathbb{F}_q$ then each set $E_1(a_1)$ and $E_2(a_2)$ contains at least $q(q-1)+1$ points. The total number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is (q^2+q+1), hence

$$2(q(q-1)+1) \leq (q^2+q+1)$$

which implies $q \leq 2$. It completes the proof for finite fields k.

For infinite k, let $\mathfrak{l}_1, \mathfrak{l}_2$ be lines through $\mathfrak{q}_2 \in E_2$ which do not pass through $\mathfrak{q}_1$ and which intersect $\mathbb{A}^1_{\lambda_1}$. These lines intersect all lines through $\mathfrak{q}_1 \in E_1$. Note that all those intersection points in $E_1 \setminus \Pi_{\lambda_1}$ are different for $\mathfrak{l}_1 \neq \mathfrak{l}_2$.

The generic value on $\mathfrak{l}_1, \mathfrak{l}_2$ is a_2 and hence at all but at most 4 lines $\mathfrak{l}$ in E_1 through $\mathfrak{q}_1$ we have at least 2 points $\mathfrak{l} \cap \mathfrak{l}_i$ with value a_2 which contradicts the fact that generic value on $\mathfrak{l}$ is a_1. □

This concludes the proof of Theorem 6.

□

Corollary 8. (1) *Theorem 6 holds also $\mathbb{P}^n(\mathbb{F}_2)$ if f takes at least $(n+1)$ distinct values.*
(2) *There is only one two-valued map on $\mathbb{P}^2(\mathbb{F}_2)$, modulo projective transformations, which is not a flag map.*

Proof. The first statement follows by induction on dimension, the case of $\mathbb{P}^2(\mathbb{F}_2)$ clear by Lemma 7. The second statement follows by direct verification. □

Lemma 9. *Let*

$$f : \mathbb{P}(K) = K^\times / k^\times \to A$$

be a homomorphism whose restriction to every line is a flag map and such that there exists a plane $\Pi = \Pi(1, x, y)$, with $f(x), f(y) \neq 1$, and f_Π not a flag map. Then

$$f(x) = f(y) \quad \text{and} \quad f(x)^2 = 1.$$

In particular, if f is not a flag map, then $k = \mathbb{F}_2$ and f^2 is a flag map.

Proof. Let $\Pi := \Pi(1, x, y)\rangle \subset \mathbb{P}(K)$ such that f_Π is not a flag map. Changing x, y by projective transformations and division by an element we can assume that f takes two values $1, a$ on Π, with

$$f(1) = f(x+1) = f(y+1) = f(x+y) = 1$$

and

$$f(x) = f(y) = f(x+y+1) = a.$$

On $\mathfrak{l}(xy, x+y+1)$, we have

$$f(xy) = a^2, \quad f(xy+x+y+1) = 1, \quad f(x+y+1) = a$$

and hence three values. Since $a^2 \neq a$, by assumption, f is not flag on $\mathfrak{l}(xy, x+y+1)$. $\square$

Lemma 10. *Assume that the two-torsion part* $A[2] \subseteq A$ *is nontrivial. Consider the composition*

$$f_2 : \mathbb{P}(K) \xrightarrow{f} A \xrightarrow{r_2} A/A[2],$$

with r_2 *the projection. Then* f_2 *is a flag map on every plane* $\Pi \subset \mathbb{P}(K)$*.*

Proof. If f is a flag map on $\Pi(1, x, y)$, then f_2 is also flag. If f is not a flag map, then we apply Lemma 9: f takes only two values, 0 or 1, and $f(x)^2 = 1$, thus $f(x) = 1$.

In particular, $f_2 \equiv 1$ on $\Pi(1, x, y)$ and hence is a flag map. Thus f_2 is a flag map on every plane, and hence a flag map. $\square$

To summarize, if $A \neq A[2]$ then f determines a valuation ν. If $A = A[2]$, then f is trivial on some subfield $K' \subset K$ such that $K|K'$ is a purely inseparable extension of exponent 2.

3. $\mathbb{Z}_{(p)}$-LATTICES

Let p be a prime number and $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ the set of rational numbers with denominator coprime to p. A $\mathbb{Z}_{(p)}$-lattice, or simply, a lattice $B \subset \mathbb{Q}^{n+1}$ is a $\mathbb{Z}_{(p)}$-submodule such that $B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = \mathbb{Q}^{n+1}$. Given a lattice $B \subset \mathbb{Q}^{n+1}$ and an element $x \in \mathbb{Q}^{n+1} \setminus 0$, there exists an element $x_B \in B \setminus pB$ such that x and x_B define the same point in $\mathbb{P}^n(\mathbb{Q})$, this element is unique in $B \setminus pB$, modulo scalar multiplication by $\mathbb{Z}_{(p)}^\times$. Lattices $B, B' \subset \mathbb{Q}^{n+1}$ are called equivalent if $B = a \cdot B'$, for some $a \in \mathbb{Q}^\times$.

In this section, we consider the maps

$$f : (\mathbb{Q}^{n+1} \setminus 0) \to A,$$

which are invariant under scalar multiplication by $\mathbb{Q}^\times$; and we use the same notation for the induced map

$$f : \mathbb{P}^n(\mathbb{Q}) \to A.$$

We say that f is *induced from* $\mathbb{P}^n(\mathbb{Z}/p)$ *via a lattice* B if there exists a map

$$\bar{f} : \mathbb{P}^n(\mathbb{Z}/p) \to A,$$

such that

$$f(x) = (\bar{f} \circ \rho_B)(x_B), \quad \text{for all} \quad x \in \mathbb{P}^n(\mathbb{Q}),$$

where

$$\rho_B : (B \setminus pB) \to (B/pB) \setminus 0 \to \mathbb{P}^n(\mathbb{Z}/p).$$

This is well-defined since ρ_B is invariant under $\mathbb{Z}_{(p)}^\times$. Any such lattice will be called *f-compatible*, or simply *compatible*. If f is induced from $\mathbb{P}^n(\mathbb{Z}/p)$ via a lattice B, then it is also induced via any equivalent lattice.

Any sublattice of $\mathbb{Q}^n$ is compatible with a constant function. However, if f takes at least two values then the set of f-compatible lattices is much smaller. Note that equivalence classes lattices in $\mathbb{Q}^2$ are naturally parametrized by a p-tree (a tree where each vertex has $(p+1)$ outgoing edges).

Lemma 11. *If f is a $\mathbb{Q}^\times$-homogeneous function on $\mathbb{Q}^2\setminus 0$ and $f(x)\neq f(y)$ then the set of f-compatible lattices consists of $\mathbb{Z}_{(p)}$-lattices generated by $p^{m_x}x, p^{m_y}y$, with $m_x, m_y\in\mathbb{Z}$.*

Proof. Let B be f-compatible and consider the projection ρ_B. Then, for $x\in\mathbb{Q}^2\setminus 0$ there is a unique $m_x\in\mathbb{Z}$ such that $p^{m_x}x$ is a generator of B, i.e., $p^{m_x}x\in B\setminus pB$. We have $\rho_B(p^{m_x}x)=\bar{x}\in\mathbb{P}^1(B/pB)$. Consider $y\in\mathbb{Q}^2\setminus 0$ and $p^{m_y}y\in B\setminus pB$ with $\rho_B(p^{m_y}y)=\bar{y}\in\mathbb{P}^1(B/pB)$. Then $\rho_B(p^{m_y}y)\neq\rho_B(p^{m_x}x)$ since f is induced from $\bar{f}$ on $\mathbb{P}^1(B/pB)$ and $f(x)\neq f(y)$. This implies that $p^{m_x}x$ and $p^{m_y}y$ generate B. □

In the discussion below, we use projective and affine geometry. The following lemma connects these concepts.

Lemma 12. *Assume that $f:\mathbb{P}^1(\mathbb{Q})\to A$ is induced from a nonconstant map $\bar{f}:\mathbb{P}^1(\mathbb{Z}/p)\to A$, via some lattice.*

1. *If $\bar{f}$ is a flag map, then there are exactly two equivalence classes of f-compatible lattices $B_1, B_2\subset\mathbb{Q}^2$.*
2. *If $\bar{f}$ is not a flag map, then there is exactly one equivalence class of f-compatible lattices $B\subset\mathbb{Q}^2$.*

Proof. By assumption, f is induced via some ρ_B. Fix generators $x,y\in B$ such that $f(y)\neq f(x)$, in particular $\rho_B(x_B)\neq\rho_B(y_B)\in\mathbb{P}^1(\mathbb{Z}/p)$. We have

$$f(y+pB)=f(y)\text{ and }f(x+pB)=f(x)\neq f(y).$$

Any lattice $B'\subset\mathbb{Q}^2$ is equivalent to a lattice with x as a generator. Since $B'/\mathbb{Z}_{(p)}\cdot x\simeq\mathbb{Z}_{(p)}$, B' is one of the following: $B_i:=\langle x,p^iy\rangle$, for some $i\in\mathbb{Z}$. If f is induced from B_i, for some $i<-1$, then

$$f(x+p(p^iy))=f(x)\neq f(y)\text{ and }f(x+p(p^iy))=f(p^{-i-1}x+y)=f(y),$$

a contradiction. The same argument yields a contradiction when $i>1$. Thus $i=1,0,$ or -1.

Analysis of values of $\bar{f}$ at other points of $\mathbb{P}^1(\mathbb{Z}/p)$ leads to further restrictions. We have the following cases:

(1) $\bar{f}$ is constant on $\mathbb{P}^1(\mathbb{Z}/p) \setminus \rho_B(y_B)$.
(2) $\bar{f}$ is not constant on the complement to a point in $\mathbb{P}^1(\mathbb{Z}/p)$.

In Case (1), $f(x+y) = f(x)$, excluding $i = 1$. Then we have exactly two lattices B_0, B_{-1}, such that f is induced from these (or equivalent) lattices.

In Case (2), if f is induced from B_1 then

$$f(\kappa x + y) = f(p\kappa x + p^{-1}y) = f(y), \quad \text{for any} \quad \kappa \in \mathbb{Z}_{(p)},$$

and hence $\bar{f}$ is constant on $\mathbb{P}^1(\mathbb{Z}/p) \setminus \rho_B(x_B)$, contradicting the second condition. A similar argument works for B_{-1}. Thus there is only one compatible lattice $B_0 = B$, modulo equivalence. □

A similar analysis holds for f-compatible lattices in $\mathbb{Q}^n$, for arbitrary n. For $x \in \mathbb{Q}^n \setminus 0$ we let $\langle x \rangle$ be the ray consisting of its nonzero multiples. Then, for any sublattice B, the image of $\langle x \rangle_B$ in $\mathbb{P}^{n-1}(\mathbb{Z}/p)$ is well-defined. However, it may happen that for some B the corresponding images lie in a proper subspace of $\mathbb{P}^{n-1}(\mathbb{Z}/p)$ while for another f-compatible lattice they span the whole $\mathbb{P}^{n-1}(\mathbb{Z}/p)$.

Lemma 13. *Assume that f is induced from $\bar{f}$ via some lattice B and that the images $\langle x_1 \rangle_B, \langle x_2 \rangle_B, \dots, \langle x_n \rangle_B$ span $\mathbb{P}^{n-1}(\mathbb{Z}/p)$. Then B is generated by $p^{i_1}x_1, \dots, p^{i_n}x_n$, for some $i_1, \dots, i_n$.*

Lemma 14. *Assume that $f : \mathbb{P}^2(\mathbb{Q}) \to A$ satisfies the following:*

(1) *f takes three values;*
(2) *f takes at most two values on every line $\mathfrak{l} \subset \mathbb{P}^2$;*
(3) *on every $\mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$, f is induced from a flag map on $\mathbb{P}^1(\mathbb{Z}/p)$, via $\rho_{B'}$, for some lattice $B' \subset \mathbb{Q}^2$.*

Then there are exactly three equivalence classes of lattices $B_i \subset \mathbb{Q}^3$ such that f is induced from a flag map $\bar{f} : \mathbb{P}^2(\mathbb{Z}/p) \to A$, via ρ_{B_i}, $i = 1, 2, 3$.

Proof. It follows from Proposition 4, applied to $k = \mathbb{Q}$ (see also [12] or [4]). The first two conditions imply that there exists a lattice $B \subset \mathbb{Q}^3$ such that f is induced from some We conclude that $\bar{f}$ is a flag map, with 3 distinct values. Hence

$$\mathbb{P}^2(\mathbb{Q}) = S_1 \sqcup S_2 \sqcup S_3,$$

with S_1 the preimage of an affine plane in $\mathbb{P}^2(\mathbb{Z}/p)$, S_2 an affine line, and S_3 a point in $\mathbb{P}^2(\mathbb{Z}/p)$, and f is constant on these sets.

Thus, for any $B' \subset \mathbb{Q}^3$ such that f is induced from $\mathbb{P}^2(\mathbb{Z}/p)$ via $\rho_{B'}$, the restriction of f to any $(\mathbb{Q}^2 \setminus 0) \subset (\mathbb{Q}^3 \setminus 0)$ is induced from a flag map on $\mathbb{P}^1(\mathbb{Z}/p)$. Hence f is also induced from a flag map, via $\rho_{B'}$. On the other hand, in coordinates x_1, x_2, x_3, we have

$$S_1 = \{x_1 \neq 0\}, \quad S_2 = \{x_1 = 0, x_2 \neq 0\}, \quad S_3 = \{x_1 = x_2 = 0, x_3 \neq 0\},$$

and the only possible coordinates compatible with the structures on all $\mathbb{P}^1(\mathbb{Q})$ are

$$x_1, \frac{x_2}{p}, \frac{x_3}{p}, \quad x_1, x_2, \frac{x_3}{p}, \text{ and } \quad x_1, x_2, x_3.$$

Indeed, consider lattices B_1, B_2, B_3 as above and assume that there exists another sublattice $B' \subset \mathbb{Q}^3$ such that f is induced via B'. Choose $\mathbb{Q}_1^2 \subset \mathbb{Q}^3$ so that f is not constant on $\mathbb{Q}_1^2$. Then we can choose B_i so that

$$B_i \cap \mathbb{Q}_1^2 = a \cdot B' \cap \mathbb{Q}_1^2, \quad a \in \mathbb{Q}^\times,$$

since there are only two possible equivalence classes of compatible lattices in $\mathbb{Q}_1^2$. Note that there are at least two such B_i whose intersections with a complementary subspace $\mathbb{Q}_2^2$, with f nontrivial on it, are different. Thus for at least one B_i we can assume that

$$B' \cap \mathbb{Q}_j^2 = B_i \cap \mathbb{Q}_j^2, \quad j = 1, 2$$

and $B_i \cap \mathbb{Q}_j^2$ generate B_i over $\mathbb{Z}_{(p)}$. Hence B' is equal to B_i, for one of the $i = 1, 2, 3$. □

4. A result from field theory

Let

$$k \subseteq \tilde{k} \subseteq \tilde{k}_a \subseteq K$$

be an extension of fields. We say that $x_1, x_2 \in K^\times/k^\times$ are algebraically dependent with respect to $\tilde{k}$ if they satisfy Equation (4) from the Introduction; in this case, we write $x_1 \sim_{\tilde{k}} x_2$, or simply $x_1 \sim x_2$. We record the following obvious properties of this equivalence relation:

(AI) If $x_1 \sim_{\tilde{k}} x_2$, $x_1/x_2 \notin \tilde{k}_a^\times/k^\times$, and $x \not\sim_{\tilde{k}} x_1$, then $x_1/x \not\sim_{\tilde{k}} x_2/x$.

(AG) For all $x_1 \in K^\times/k^\times \setminus \tilde{k}_a^\times/k^\times$, the set of x_2 such that the closure of $\tilde{k}_a(x_1)$ in K coincides with the closure of $\tilde{k}_a(x_2)$, together with $(\tilde{k}_a^\times/k^\times)$, forms a subgroup of $K^\times/k^\times$.

Lemma 15. *Let $K|k$ and $L|l$ be field extensions, ν a valuation of K, and*

$$\psi : K^\times/k^\times \to L^\times/l^\times \tag{10}$$

a homomorphism, such that its restriction to $\mathfrak{o}_{K,\nu}^\times/\mathfrak{o}_{k,\nu}^\times$ factors as

$$\mathfrak{o}_{K,\nu}^\times/\mathfrak{o}_{k,\nu}^\times \twoheadrightarrow \boldsymbol{K}_\nu^\times/\boldsymbol{k}_\nu^\times \xrightarrow{\psi_\nu} L^\times/l^\times. \tag{11}$$

Assume that ψ_ν preserves algebraic dependence with respect to $\boldsymbol{k}_\nu$ and l. Then ψ also preserves algebraic dependence with respect to k and l.

Proof. Let $k(x) \subset K$ be a purely transcendental extension and

$$E = \overline{k(x)} \subset K$$

its algebraic closure in K. We claim that the restriction of ψ to $E^\times/k^\times$ preserves algebraic dependence. There are two cases:

Case 1. $\nu(k^\times) = \nu(E^\times)$. Then

$$E^\times = \mathfrak{o}_{E,\nu}^\times \cdot k^\times.$$

Since ψ_ν preserves algebraic dependence with respect to $\boldsymbol{k}_\nu$ and l, the claim follows.

Case 2. $\nu(k^\times) \subsetneq \nu(E^\times)$. Then $\nu(E^\times)/\nu(k^\times)$ has $\mathbb{Q}$-rank 1, i.e., for $y, z \in E^\times$ with nonzero $\nu(y), \nu(z) \in \nu(E^\times)/\nu(k^\times)$ there are nonzero $n_y, n_z \in \mathbb{Z}$ such that $n_y\nu(y) = n_z\nu(z)$. Indeed, y, z define a finite algebraic extension $k_{y,z}(x)|k(x)$, hence ν is nontrivial on $k(x)$, and the group

$$\nu(k_{y,z}(x)^\times)/\nu(k(x)^\times)$$

is finite. Let $g \in k(x)^\times$ be such that the image of $\nu(g)$ in $\nu(E^\times)/\nu(k^\times)$ is infinite. Then for any $\sum_{i=0}^n a_i g^i$, with $a_i \in k$,

$$\nu(\sum_{i=0}^n a_i g^i) = \min_i(\nu(a_i g^i)),$$

since none of the monomials $a_i g^i$ have the same value under ν. Thus,

$$\nu(k(g)^\times) = \nu(k^\times) \times \langle \nu(g) \rangle.$$

The extensions $k_{y,z}(x)|k(x)$ and $k(x)|k(g)$ are finite, thus

$$\nu(k_{y,z}(x)^\times)/(\nu(k^\times) \times \langle \nu(g) \rangle)$$

is also finite, which implies the result for $\nu(E^\times)$. Since $\psi(k^\times) = 1$, $\psi(k_{y,z}(x)^\times)$ is the product of a finite group and $\mathbb{Z}$. In particular, $\psi(k_{y,z}(x)^\times)$ consists of algebraically dependent elements. Since E is a union of subfields $k_{y,z}(x)$, the same holds for $E^\times$.

Thus $\psi(E^\times/k^\times)$ coincides with the image of $\nu(E^\times)/\nu(k^\times)$. Since all elements in $\nu(E^\times)/\nu(k^\times)$ have the same powers, i.e., the $\mathbb{Q}$-rank of this group is at most one, we see that the lifts of elements in $\psi(E^\times)$ to $L^\times$ are algebraically dependent over l. $\square$

5. Restriction to planes

Here we study the restrictions of homomorphisms

$$\psi : \mathbb{P}(K) = K^\times/k^\times \to A := L^\times/\tilde{l}^\times,$$

satisfying the assumptions of Theorem 2, to projective planes $\Pi \subset \mathbb{P}(K)$.

Proposition 16. *Let $\Pi := \Pi(1, x, y) \subset \mathbb{P}(K)$ be a projective plane such that $\psi(x) \not\sim \psi(y)$. Then one of the following holds:*

(a) *ψ_Π is injective.*
(b) *There exists a line $\mathfrak{l} \subset \Pi$ such that ψ_Π is constant on $\Pi \setminus \mathfrak{l}$.*
(c) *There exists a point $\mathfrak{q} \in \Pi$ such that ψ_Π is constant on $\mathfrak{l} \setminus \mathfrak{q}$, for every $\mathfrak{l} \subset \Pi$ passing through $\mathfrak{q}$.*
(d) *$k = \mathbb{Q}$, ψ_Π is induced from*

$$\bar{\psi}_\Pi : \mathbb{P}^2(\mathbb{Z}/p) \to A,$$

via a lattice $B \subset \mathbb{Q}^3$, and $\bar{\psi}_\Pi$ is of type (a), (b), or (c).

Proof. Assume that ψ_Π is not injective: there are distinct $x_1, x_2 \in \Pi$, with $\psi(x_1) = \psi(x_2) \neq 1$. Consider

$$\Pi_1 := x_1^{-1} \cdot \Pi = \Pi(1, 1/x_1, y/x_1),$$

since $\psi(y) \not\sim \psi(1/x_1)$, Π_1 satisfies the conditions of the theorem; if it holds for Π_1, then it holds for the initial Π. Thus we may assume that

$$S_1 := \{x' \in \Pi \mid \psi(x') = 1\} \tag{12}$$

contains at least two elements. Consider the map

$$\psi_\sim : \mathbb{P}(K) \to A_\sim,$$

with values in dependency classes:

- $\psi_\sim(x') = 1$ if $\psi(x') = 1$,
- $\psi_\sim(x') = \psi_\sim(x'')$ iff $\psi(x'), \psi(x'') \neq 1$ and $\psi(x') \sim \psi(x'')$.

We record the properties of $\psi_\sim$:

(TI) For every $\mathfrak{l} \subset \Pi$ with $\mathfrak{l} \cap S_1 = \emptyset$, we have

$$\{\psi_\sim(x') \mid x' \in \mathfrak{l}\} = \{\psi_\sim(x'') \mid x'' \in \Pi \setminus S_1\},$$

in particular, $\psi(\mathfrak{l})$ has algebraically independent elements.

(TC) For every $\mathfrak{l} \subset \Pi$ with $\mathfrak{l} \cap S_1 \neq \emptyset$, $\psi_\sim$ is constant on $\mathfrak{l} \setminus (\mathfrak{l} \cap S_1)$.

Property (AI) from Section 4 relates $\psi_\sim$ and ψ.

Lemma 17. *If $\mathfrak{l} \cap S_1 = \emptyset$ and $x', x'' \in \mathfrak{l}$ are such that $\psi(x') \sim \psi(x'')$, then $\psi(x') = \psi(x'')$.*

Proof. There is a $z \in \mathfrak{l}$ with $\psi(z) \not\sim \psi(x'), \psi(x'')$. Since $z^{-1} \cdot \mathfrak{l} \cap S_1 \neq \emptyset$, all values of ψ on $\mathfrak{l}(x'/z, x''/z) \setminus 1$ are algebraically dependent (here we use that $\tilde{l}_a = \tilde{l}$). By (AI), if $\psi(x') \neq \psi(x'')$, then $\psi(x')/\psi(z) \not\sim \psi(x'')/\psi(z)$, a contradiction. □

Lemma 18. *Let $\mathfrak{l}, \mathfrak{l}' \subset \Pi$ be disjoint from S_1, put $z := \mathfrak{l} \cap \mathfrak{l}'$, and assume that there exist $x \in \mathfrak{l}$ and $x' \in \mathfrak{l}'$ such that*

$$\psi(x) \sim \psi(x'), \psi(x) \neq \psi(x'), \quad \text{and} \quad \psi(x), \psi(x') \not\sim \psi(z).$$

Let $y \in \mathfrak{l}$ and $y' \in \mathfrak{l}'$ be such that $\psi(y) \sim \psi(y')$. Then either

- *$\psi(y) \neq \psi(y')$, or*
- *$\psi(y) = \psi(y') = \psi(z)$.*

Proof. Assume that

$$\psi(y) = \psi(y') \neq \psi(z),$$

then

$$\frac{\psi(x)}{\psi(z)} \sim \frac{\psi(y)}{\psi(z)}$$

and

$$\frac{\psi(x')}{\psi(z)} \sim \frac{\psi(y')}{\psi(z)},$$

by (TC), by the same argument as in Lemma 17. Finally, by (AI),

$$\frac{\psi(x)}{\psi(z)} \sim \frac{\psi(x')}{\psi(z)}$$

is not possible. □

Let $\{T_j\}_{j \in J}$ be the set of intersections of algebraic dependency classes in $\mathbb{P}(K)$ with Π. Split $J = J_2 \sqcup J_3$ and consider the decomposition

(13) $\Pi = S_1 \sqcup S_2 \sqcup S_3, \quad \text{with} \quad S_1 = T_1, S_2 = \sqcup_{j \in J_2} T_j, \quad S_3 = \sqcup_{j \in J_3} T_j,$

(here S_1 is the same set as in (12)).

For any such decomposition, the induced map

$$\Psi = \Psi_\Pi : \Pi \to \{1, 2, 3\}$$

factors through $\psi_\sim$ and satisfies the conditions of Proposition 4. Thus Ψ is induced from a trivial coloring, with S_1 not depending on the decomposition. Since there exist lines disjoint from S_1, and S_1 contains at least two points, it follows that either

- (B) $S_1 = \Pi \setminus \mathfrak{l}$, for some $\mathfrak{l} \subset \Pi$, and we are in Case (b), or
- (C) $S_1 = \cup_{i \in I}(\mathfrak{l}_i \setminus \mathfrak{q})$, for some $\mathfrak{q} \in \Pi$ and $\mathfrak{l}_i$ through $\mathfrak{q}$, and we are in Case (c), as is proved below in Lemma 19, or
- (D) $k = \mathbb{Q}$, and Ψ is induced from a trivial coloring on $\mathbb{P}^2(\mathbb{Z}/p)$.

Note that in Case (B), $\psi \equiv 1$ on the affine plane $\Pi \setminus \mathfrak{l}$.

Lemma 19. *In case* (C), ψ *is constant on an affine plane, or on* $\mathfrak{l}\setminus\mathfrak{q}$, *for all lines* $\mathfrak{l}$ *passing through* $\mathfrak{q}$.

Proof. Consider $x \in \Pi \setminus (S_1 \cup \mathfrak{q})$ and lines $\mathfrak{l}$ containing x but not $\mathfrak{q}$. Then $\psi_\sim \equiv \psi_\sim(x)$ on $\mathfrak{l}\setminus(\mathfrak{l}\cap S_1)$, by (TI). Since S_1 is not an affine plane, there is an $x' \in \Pi \setminus (S_1 \cup \mathfrak{l}(x,\mathfrak{q}))$. We have $\psi_\sim(x) = \psi_\sim(x')$. The union of lines $\mathfrak{l} \subset \Pi, \mathfrak{q} \notin \mathfrak{l}$, through x, x', is equal to $\Pi\setminus\mathfrak{q}$. Thus $\psi_\sim$ takes only three values $\{1, \psi(x), \psi(\mathfrak{q})\}$ and is constant on $\Pi\setminus(S_1\cup\mathfrak{q})$. Lemma 17, applied to $\mathfrak{l}$ through $\mathfrak{q}$, implies that ψ is constant on $\mathfrak{l}\setminus\mathfrak{q}$. ☐

We are left with Case (D), when Ψ is induced via some

$$\rho : \Pi = \mathbb{P}^2(\mathbb{Q}) \to \mathbb{P}^2(\mathbb{Z}/p)$$

from a trivial coloring

$$\mathfrak{c} : \mathbb{P}^2(\mathbb{Z}/p) \to \{1, 2, 3\},$$

in the sense of Proposition 4. Put

$$\bar{S}_i = \mathfrak{c}^{-1}(i), \quad i = 1, 2, 3.$$

Note that S_1 is a finite union of subsets $\mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}$ and does not contain a complete line $\mathfrak{l}$. Consider shifts $\Pi_z := z^{-1} \cdot \Pi$, for $z \in \Pi$. The shift from Π to Π_z changes algebraically dependent subsets. Note that Π_z contains 1 and S_1 contains $\mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}$, by assumption of case (D), thus on lines $\mathfrak{l}$ through 1 all elements in $\mathfrak{l}\setminus 1$ are algebraically dependent. If there are at least two elements $x/z, y/z$ with $\psi(x/z) \not\sim \psi(y/z)$ then we have a splitting into S_1, S_2, S_3, and since S_1 contains $\mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}$ we can proceed by induction.

Lemma 20. *For every* $z \in \Pi$, *the restriction of* $\psi_\sim$ *to* Π_z *is induced from* $\mathbb{P}^2(\mathbb{Z}/p)$.

Proof. We subdivide (D) into subcases:

- (D1) For every z and every splitting $\Pi_z = S_{1,z} \sqcup S_{2,z} \sqcup S_{3,z}$, where $S_{2,z}, S_{3,z}$ are unions of algebraic dependency classes, the set $\bar{S}_{1,z} \subset \mathbb{P}^2(\mathbb{Z}/p)$ is either a point, an affine line, or an affine plane.
- (D2) Otherwise: for some Π_z this is not the case.

First we treat (D1). Fix Π_z and a decomposition $\Pi_z = S_{1,z}\sqcup S_{2,z}\sqcup S_{3,z}$; we have

$$\bar{\Psi} : \mathbb{P}^2(\mathbb{Z}/p) \to \{1, 2, 3\},$$

and

$$\mathbb{P}^2(\mathbb{Z}/p) = \sqcup_{i=1}^3 \bar{S}_{i,z}, \quad S_{i,z} = \rho^{-1}(\bar{S}_{i,z}) \subset \Pi.$$

By assumption (D1), we have 3 cases.

- $\bar{S}_{1,z} = \bar{\mathfrak{q}}$, for some $\bar{\mathfrak{q}} \in \mathbb{P}^2(\mathbb{Z}/p)$. For $x \in \Pi_z \setminus S_{1,z}$ and $\mathfrak{l} = \mathfrak{l}(\mathfrak{q}, x)$, with $\rho(\mathfrak{q}) = \bar{\mathfrak{q}}$, $\psi_\sim$ is constant on $\mathfrak{l} \setminus (\mathfrak{l} \cap S_{1,z})$, by (TC). Apply this to all $\mathfrak{l}(\mathfrak{q}, x_1)$, where x_1 runs over $S_{1,z}$, to conclude that $\psi_\sim$ is constant on preimages of affine lines $(\bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}})$, with $\bar{\mathfrak{q}} \in \bar{\mathfrak{l}}$, hence is induced from $\mathbb{P}^2(\mathbb{Z}/p)$.
- $\bar{S}_{1,z} = \bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}}$, for some $\bar{\mathfrak{l}} \subset \mathbb{P}^2(\mathbb{Z}/p)$ and $\bar{\mathfrak{q}} \in \bar{\mathfrak{l}}$. Then $\bar{S}_{1,z}, \bar{S}_{2,z}$ and $\bar{S}_{3,z}$ form a flag on $\mathbb{P}^2(\mathbb{Z}/p)$: all points projecting to $\mathbb{P}^2(\mathbb{Z}/p) \setminus \bar{\mathfrak{l}}$ belong to the same algebraic dependency class because each pair of such points can be connected by a pair of lines which intersect $S_{1,z}$. Lemma 14 reduces the proof to the previous case, after changing to a different $\psi_\sim$-compatible lattice.
- $\bar{S}_{1,z} = \mathbb{P}^2(\mathbb{Z}/p) \setminus \bar{\mathfrak{l}}$, for some line $\bar{\mathfrak{l}} \in \mathbb{P}^2(\mathbb{Z}/p)$. This reduces to the case $\bar{S}_{1,z} = \bar{\mathfrak{q}}$.

We pass to (D2) and fix a plane Π_z, with a splitting

$$\Pi_z = S_{1,z} \sqcup S_{2,z} \sqcup S_{3,z},$$

violating (D1). Note that Π_z contains points $1, 1/z, x/z$ with $\psi(1/z) \not\sim \psi(x/z)$. We also know that the subset $S_1/z \subset \Pi_z$, with S_1 defined in (13), is a finite union of subsets projectively equivalent to $\mathbb{Z}_{(p)} + \mathbb{Z}_{(p)} \subset \mathbb{P}^2(\mathbb{Q})$. By Proposition 4, we have an induction of Ψ_z from the trivial coloring on $\mathbb{P}^2(\mathbb{Q}_\nu)$, for some valuation ν on $\mathbb{Q}$, and we obtain that ν is nontrivial on $\mathbb{Q}$, i.e., corresponds to some prime number. Since S_1/z is contained in either $S_{2,z}$ or $S_{3,z}$, the corresponding prime equals p. Thus Ψ_z is induced from a trivial 3-coloring of $\mathbb{P}^2(\mathbb{Z}/p)$.

Now, we may assume that $\bar{S}_{1,z}$ is a union of more than one subset of type $\mathfrak{l}_i \setminus \mathfrak{q} \subset \mathbb{P}^2(\mathbb{Z}/p)$ (otherwise, we are in Case (D1)). Note that one of the subsets $\bar{S}_{i,z}, i = 2, 3$ is $\bar{\mathfrak{q}}$ and the complement of all such is $S_{3,z}$.

Then there exist a point $\bar{\mathfrak{q}} \in \mathbb{P}^2(\mathbb{Z}/p)$ and a set $\{\bar{\mathfrak{l}}_i\}_{i \in I}$ of at least two lines passing through $\bar{\mathfrak{q}}$ such that $\bar{S}_{1,z} = \cup_{i \in I}(\bar{\mathfrak{l}}_i \setminus \bar{\mathfrak{q}})$, since we are in the case (D2), by assumption. Moreover, we may assume that $\bar{S}_{2,z} = \bar{\mathfrak{q}}$, then $\bar{S}_{3,z}$ has the same structure as $\bar{S}_{1,z}$, i.e., a union of affine lines containing $\bar{\mathfrak{q}}$ in their closure.

We claim that $\psi_\sim$ is constant on $S_{3,z}$: consider $\bar{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3' \in \bar{S}_{3,z}$ not lying on a line through $\bar{\mathfrak{q}}$. Let $\mathfrak{q}_3, \mathfrak{q}_3'$ be any points projecting to $\bar{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3'$. Since $\bar{\mathfrak{l}}(\bar{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3') \cap \bar{S}_1 \neq \emptyset$, the line $\mathfrak{l}(\mathfrak{q}_3, \bar{\mathfrak{q}}_3')$ intersects S_1, thus $\psi(\mathfrak{q}_3) \sim \psi(\mathfrak{q}_3')$. By assumption on $\bar{S}_{3,z}$, any two points in $S_{3,z}$ can be connected by a chain of such lines.

Note that $\psi_\sim$ is constant on $S_{2,z}$: consider

$$\mathfrak{q}_1, \mathfrak{q}_2 \quad \text{with} \quad \rho(\mathfrak{q}_1) = \rho(\mathfrak{q}_2) = \bar{\mathfrak{q}} \in \bar{S}_{2,z}.$$

Then $\psi(\mathfrak{q}_1) = \psi(\mathfrak{q}_2)$. Indeed, consider $\mathfrak{l}_5 = \mathfrak{l}(\mathfrak{q}_1, x_1)$ and $\mathfrak{l}_6 = \mathfrak{l}(\mathfrak{q}_2, x_2)$, where $\rho(x_i) = \bar{x}_i \in \bar{S}_1, \bar{x}_1 \neq \bar{x}_2$. Hence $\mathfrak{q}_3 := \mathfrak{l}_5 \cap \mathfrak{l}_6$ projects to $\bar{\mathfrak{q}}$. Thus

$\psi_\sim(\mathfrak{q}_1) = \psi_\sim(\mathfrak{q}_3) = \psi_\sim(\mathfrak{q}_2)$. Thus $\psi_\sim$ is constant on $S_{2,z}$, hence $\psi_\sim$ is induced from $\mathbb{P}^2(\mathbb{Z}/p)$. ☐

Now we use $\psi_\sim$ to prove the induction from $\mathbb{P}^2(\mathbb{Z}/p)$ result for ψ itself. The difference between S_1 and S_2, S_3 is that ψ is already constant on S_1 but not necessarily on S_2, S_3. We treat the cases:

(1) $\bar{S}_1 = \bar{\mathfrak{q}}, \bar{S}_2 = \bar{\mathfrak{l}} \setminus \mathfrak{q}, \bar{\mathfrak{q}} \in \bar{\mathfrak{l}}, \bar{S}_3 = \mathbb{P}^2(\mathbb{Z}/p) \setminus \bar{\mathfrak{l}}$;
(2) $\bar{S}_1 = \bar{\mathfrak{q}}, \bar{S}_2 = \cup_{i=1}^m \bar{\mathfrak{l}}_i \setminus \mathfrak{q}, m \geq 2, \bar{\mathfrak{q}} \in \bar{\mathfrak{l}}_i, \bar{S}_3 = \mathbb{P}^2(\mathbb{Z}/p) \setminus \cup \bar{\mathfrak{l}}_i$;
(3) $\bar{S}_1 = \cup_{i=1}^m \bar{\mathfrak{l}}_i \setminus \mathfrak{q}, m \geq 2, \bar{S}_2 = \bar{\mathfrak{q}}, \bar{S}_3 = \mathbb{P}^2(\mathbb{Z}/p) \setminus \cup \bar{\mathfrak{l}}_i$.

Lemma 21. *The map ψ_Π is induced from $\bar{\psi}_\Pi : \mathbb{P}^2(\mathbb{Z}/p) \to A$ which is of the type (a), (b), or (c).*

Proof. By Lemma 20, we have the following possibilities:

(1) $\psi_\sim$ is induced from a flag map on $\mathbb{P}^2(\mathbb{Z}/p)$ and we can assume that $\bar{S}_1 = \bar{\mathfrak{q}}$, by Lemma 14;
(2) $\psi_\sim$ is induced from a map on $\mathbb{P}^2(\mathbb{Z}/p)$ which is constant on affine lines $\bar{\mathfrak{l}}_i \setminus \bar{\mathfrak{q}}$, with $\bar{\mathfrak{q}} \in \bar{\mathfrak{l}}$, and $\bar{S}_1 = \bar{\mathfrak{q}}$;
(3) $\psi_\sim$ is induced from a map on $\mathbb{P}^2(\mathbb{Z}/p)$ which is constant on affine lines $\bar{\mathfrak{l}}_i \setminus \bar{\mathfrak{q}}$, with $\bar{\mathfrak{q}} \in \bar{\mathfrak{l}}$, and $\bar{S}_1$ contains $\bar{\mathfrak{l}}_i \setminus \bar{\mathfrak{q}}, i = 1, 2$.

Case (1): We may assume that $\bar{S}_3 = \mathbb{P}^2(\mathbb{Z}/p) \setminus \bar{\mathfrak{l}}$, for some $\bar{\mathfrak{l}}$ with $\bar{\mathfrak{q}} \in \bar{\mathfrak{l}}$, and $\bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}} = \bar{S}_2$. Let $\mathfrak{l}$ be disjoint from S_1 and pick two points $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{l} \cap S_3$. Since $\psi_\sim(\mathfrak{q}) = \psi_\sim(\mathfrak{q}')$ and $\mathfrak{l}$ intersects S_2, $\psi(\mathfrak{q}) = \psi(\mathfrak{q}')$, by Lemma 17. Since any two points in S_3 can be connected by a chain of lines disjoint from S_1, ψ is constant on S_3. It is also constant on $\rho^{-1}(\bar{\mathfrak{q}}_2)$, for $\bar{\mathfrak{q}}_2 \in \bar{S}_2$. Indeed, if $\mathfrak{q}_2, \mathfrak{q}_2'$ are distinct points projecting to $\bar{\mathfrak{q}}_2$ and $\mathfrak{l}, \mathfrak{l}'$ lines containing $\mathfrak{q}_2$, resp. $\mathfrak{q}_2'$, avoiding S_1 and projecting to distinct lines in $\mathbb{P}^2(\mathbb{Z}/p)$, then $\mathfrak{q}_2'' := \mathfrak{l} \cap \mathfrak{l}'$ also projects to $\bar{\mathfrak{q}}_2$. Thus $\psi(\mathfrak{q}_2) = \psi(\mathfrak{q}_2') = \psi(\mathfrak{q}_2'')$.

Case (2): $\bar{S}_1 = \bar{\mathfrak{q}}$. If $\psi_\sim$ is induced from a noninjective $\bar{\psi} : \mathbb{P}^2(\mathbb{Z}/p) \to A$, ψ is constant on the preimage of every affine line $\bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}}$, by the same analysis over a finite field.

If there exist y_1, y_2, projecting to the same points $\bar{x} \in \bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}}$, with $\psi(y_1) \neq \psi(y_2)$, let z_1, z_2 be such that $\psi_\sim(z_1) = \psi_\sim(z_2)$ but $\psi_\sim(z_i) \neq \psi_\sim(y_i)$. Consider

$$z := \mathfrak{l}(y_1, z_1) \cap \mathfrak{l}(y_2, z_2),$$

so that $\rho(z) = \bar{x}$. Then $\psi(y_1) = \psi(z) = \psi(y_2)$, by Lemma 18. Since all points over $\bar{x}$ are connected by a chain of lines of such type, ψ is constant on $\rho^{-1}(\bar{x})$.

Case (3): The argument of Case (1) shows that ψ is constant on the preimage of any affine line $\bar{\mathfrak{l}} \setminus \bar{\mathfrak{q}}$ contained in $\bar{S}_3$. Indeed, let $z_1, z_2 \in S_3$ be in the preimage of an affine line $\bar{S}_3$ and consider $\mathfrak{l} := \mathfrak{l}(z_1, z_2)$. It intersects S_2 and hence $\psi(z_1) = \psi(z_2)$. Thus ψ is induced from $\mathbb{P}^2(\mathbb{Z}/p) \setminus \mathfrak{q} = \bar{S}_1 \cup \bar{S}_3$. Let $\mathfrak{q}, \mathfrak{q}'$, projecting $\bar{\mathfrak{q}}$. Consider lines $\mathfrak{l}(\mathfrak{q}, z_1)$ and $\mathfrak{l}(q', z_2)$ with $z_i \in S_3$, which intersect in $\mathfrak{q}'', \rho(\mathfrak{q}'') = \bar{\mathfrak{q}}$. Then $\psi(\mathfrak{q}) = \psi(\mathfrak{q}'') = \psi(\mathfrak{q}')$, by Lemma 18. Since any pair of points over $\bar{q}$ can be connected by a chain of such lines, ψ is constant on $\rho^{-1}(\bar{\mathfrak{q}})$. □

This concludes the proof of Proposition 16. □

Remark 22. This Lemma is similar to [12] and [8, Lemma 13].

6. Lines of injectivity

In our analysis of the restriction $\psi_{\mathfrak{l}}$ of

$$\psi : \mathbb{P}(K) \to A = L^\times/\tilde{l}^\times$$

to lines $\mathfrak{l} = \mathfrak{l}(1, x) \subset \mathbb{P}(K)$, we distinguish the following possibilities:

- $\psi_{\mathfrak{l}}$ is not induced from a map $\bar{\psi}_{\mathfrak{l}} : \mathbb{P}^1(\mathbb{Z}/p) \to A$ and $\psi_{\mathfrak{l}}$ is:
  - (I) injective
  - (N) not injective and nonflag
  - (F) a nonconstant flag map
- $\psi_{\mathfrak{l}}$ is induced from $\bar{\psi}_{\mathfrak{l}} : \mathbb{P}^1(\mathbb{Z}/p) \to A$ and $\bar{\psi}_{\mathfrak{l}}$ is
  - ($\bar{\text{I}}$) injective
  - ($\bar{\text{N}}$) not injective and nonflag
  - ($\bar{\text{F}}$) a nonconstant flag map
- (C) $\psi_{\mathfrak{l}}$ is constant

Definition 23. Let $\mathfrak{u} \subset \mathbb{P}(K)$ be the union of all lines through 1, on which ψ is injective, and put

$$\mathfrak{U} := \{xy \mid x, y \in \mathfrak{u}\} \subseteq \mathbb{P}(K).$$

Lemma 24. *If $\psi(\mathfrak{u})$ contains at least two algebraically independent elements, then $\mathfrak{U}$ is a group.*

Proof. Clearly, $\mathfrak{u}$ and $\mathfrak{U}$ contain $1 \in K^\times/k^\times$. If $x \in \mathfrak{U}$ then $x^{-1} \in \mathfrak{U}$, by the injectivity of ψ on $\mathfrak{l}(1, x^{-1})$. Furthermore,

$$xy^{-1} \in \mathfrak{u}, \text{ for all } x, y \in \mathfrak{u} \text{ such that } \psi(x) \neq \psi(y). \tag{14}$$

Indeed, if $\psi(x) \not\sim \psi(y)$, then ψ is injective on $\Pi(1, x, y)$, by Proposition 16, and in particular on $\mathfrak{l}(x, y) = y \cdot \mathfrak{l}(1, xy^{-1})$; thus, $xy^{-1} \in \mathfrak{u}$.

If $\psi(x) \sim \psi(y)$, but are not equal in A, take $z \in \mathfrak{u}$ such that $\psi(x) \not\sim \psi(z)$. Then $x/z, y/z \in \mathfrak{u}$, as above. Since $\psi(x/z) \not\sim \psi(y/z)$, the same argument shows that $(x/z)/(y/z) = xy^{-1} \in \mathfrak{u}$, proving (14).

To show that $\mathfrak{U}$ is multiplicatively closed, it suffices to check that for every $x_1, x_2, x_3 \in \mathfrak{u} \setminus \{1\}$ there exist $s_1, s_2 \in \mathfrak{u}$ with $x_1x_2x_3 = s_1s_2$. Note that $\psi(x_ix_j) \neq 1$ for some $1 \leq i < j \leq 3$. (Otherwise,

$$\psi(x_1x_2) = \psi(x_1x_3) = \psi(x_2x_3) = 1,$$

and therefore,

$$\psi(x_1^2) = \psi((x_1x_2)(x_1x_3)/(x_2x_3)) = 1,$$

so $\psi(x_1) = 1$.) Then, by (14), $x_ix_j \in \mathfrak{u}$, so we can take $s_1 := x_ix_j$ and $s_2 := x_t$, where $\{i, j, t\} = \{1, 2, 3\}$. □

Definition 25. Let $\bar{\mathfrak{u}} \subset \mathbb{P}(K)$ be the union of all lines $\mathfrak{l}$ through 1, such that the restriction of ψ to $\mathfrak{l}$ is induced via an injective map

$$\bar{\psi}_{\mathfrak{l}} : \mathbb{P}^1(\mathbb{Z}/p) \to A,$$

and put

$$\bar{\mathfrak{U}} := \{xy \mid x, y \in \bar{\mathfrak{u}}\} \subseteq \mathbb{P}(K).$$

Lemma 26. *If $\psi(\bar{\mathfrak{u}})$ contains at least two algebraically independent elements, then $\bar{\mathfrak{U}}$ is a group.*

Proof. The proof follows the same steps as the proof of Lemma 24. □

Lemma 27. *Assume $\mathbb{P}(K)$ contains lines of type* (I) *and one of the types*

$$(\mathrm{N}), (\bar{\mathrm{I}}), (\bar{\mathrm{N}}), \quad \textit{or} \quad (\bar{\mathrm{F}}). \tag{15}$$

Then there exists a one-dimensional subfield $E \subset L$ such that for all lines $\mathfrak{l} \subset \mathbb{P}(K)$ of type (I), (N), $(\bar{\mathrm{I}})$, $(\bar{\mathrm{N}})$, *or* $(\bar{\mathrm{F}})$ *we have*

$$\psi(\mathfrak{l}) \subset E^\times/\tilde{l}^\times.$$

In particular, if $\psi(\mathfrak{u})$ contains algebraically independent elements, lines of type (N), $(\bar{\mathrm{I}})$, $(\bar{\mathrm{N}})$, *and* $(\bar{\mathrm{F}})$ *do not exist.*

Proof. Let $\mathfrak{l} = \mathfrak{l}(1, y)$ be a line of type (I).

If there exists another line $\mathfrak{l}(1, y')$ of type (I) with $\psi(y) \not\sim \psi(y')$, i.e., $\psi(\mathfrak{u})$ contains independent elements, then lines of the listed type cannot exist, indeed, if $\mathfrak{l}(1, x)$ is of types listed in (15), we apply Proposition 16 to $\Pi = \Pi(1, x, y)$. In Case (b), the exceptional line is $\mathfrak{l}(1, y)$ and hence the restriction of ψ to any other line is either constant or of type (F), a contradiction. In Case (c), all lines are either of type (I) or (F), again a contradiction. Case (d) does not apply, since $\mathfrak{l}(1, y)$ is not induced from a map $\mathbb{P}^2(\mathbb{Z}/p) \to A$, a contradiction.

If $\psi(\mathfrak{u})$ does not contain algebraically independent elements, but one of the lines $\mathfrak{l}(1, x)$ in (15) is such that $\psi(x) \not\sim \psi(y)$, then we apply the same argument to $\Pi(1, x, y)$ and obtain the same contradiction. □

Lemma 28. *Assume $\psi(\mathfrak{u})$ contains algebraically independent elements. Consider $\mathfrak{l} := \mathfrak{l}(1, y) \not\subseteq \mathfrak{u}$ and assume that $\mathfrak{l} \cap \mathfrak{U}$ consists of at least two points $1, z'$. Then $\mathfrak{l} \cap \mathfrak{U}$ is either $\mathfrak{l}$ or $\mathfrak{l} \setminus \mathfrak{q}$, for some point $\mathfrak{q} \in \mathfrak{l}$.*

Proof. Assume that $\psi_{\mathfrak{l}}$ is not constant, e.g., $\psi(y) \neq 1$. By assumption, there is an x with $\mathfrak{l}(1, x) \subset \mathfrak{u}$ with $\psi(x) \not\sim \psi(y)$. We apply Proposition 16 to $\Pi := \Pi(1, x, y)$. We are not in Case (c) of this lemma. If we are in Case (a), then ψ is constant on $\Pi \setminus \mathfrak{l}(1, x)$, which implies that $\mathfrak{l}$ is of type (F). If we are in Case (b), then the exceptional point $\mathfrak{q} = y$, and ψ is constant, on the complement to $\mathfrak{q}$, on every line through $\mathfrak{q}$, thus $\mathfrak{l}$ is of type (F).

Put $z' = t/t'$, with $t, t' \in \mathfrak{u}$. If $\psi(t) \neq \psi(t')$ then Equation (14) implies that $z' \in \mathfrak{u}$, a contradiction. Thus $\psi_{\mathfrak{l}}$ is either constant or contains one point $y' \notin \mathfrak{U}$. In Case (a), ψ is constant on $\Pi \setminus \mathfrak{l}(1, x)$, thus identically 1 on the line $\mathfrak{l}$. In Case (b), ψ is injective on every line not containing the exceptional point $\mathfrak{q}$, in particular on $\mathfrak{l}(1, t'/t'')$, for all t'', thus $t'/t'' \in \mathfrak{u}$, thus $t'' \in \mathfrak{U}$. Taking $t'' \in \mathfrak{l} \setminus \mathfrak{q}$ we obtain the claim.

Now assume that $\psi_{\mathfrak{l}}$ is constant. We claim that $\mathfrak{l} \setminus (\mathfrak{l} \cap \mathfrak{U})$ contains at most one point. Assume otherwise. Note that ψ is injective on every line $\mathfrak{l}(u', t') \subset \Pi$, with $t' \in \Pi(1, x, y) \cap \mathfrak{u}$, $t' \neq 1$, and any point $u' \in \mathfrak{l} \cap \mathfrak{U}$. Indeed, we can represent $u' = w/w'$, with $w, w' \in \mathfrak{u}$ and with $\psi(w) = \psi(w') \not\sim \psi(t')$. Then $t'w'/w \in \mathfrak{u}$ and $\mathfrak{l}(t'w'/w, 1) \subset \mathfrak{u}$. The converse is also true, and $(\Pi \setminus \mathfrak{l}) \subset \mathfrak{u}$. Indeed, consider lines through u' which are not equal to $\mathfrak{l}$; ψ is injective on such lines.

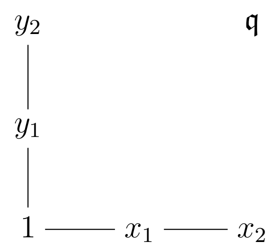
Now consider two families of lines: those passing through w (except $\mathfrak{l}$), and those throgh w' (again, except $\mathfrak{l}$). All such lines are of type (F), with generic value $\neq 1$, since ψ does not take value 1 on $\Pi \setminus \mathfrak{l}$, by Lemma 27. Consider lines $\mathfrak{l}(w, v)$ and $\mathfrak{l}(w', v)$ from these families, with $v \in (\Pi \setminus \mathfrak{l})$. The generic ψ-value on these lines is the same and is equal to $\psi(v)$. A line through u', which does not contain v cannot be of type (I), since it intersects lines $\mathfrak{l}(w, v)$ and $\mathfrak{l}(w', v)$ in distinct points, but taking the same value on these points, contradicting the established fact that such lines are of type (I). ∎

Lemma 29. *Assume $\mathbb{P}(K)$ contains lines of type $(\bar{\mathrm{I}})$ and there exist lines of type (I), or (N), or $(\bar{\mathrm{N}})$. Then $\psi(\bar{\mathfrak{u}})$ does not contain algebraically independent elements.*

Proof. Assume the contrary. Let $\mathfrak{l}(1, x)$ be a line of type (I) or (N). Then there exists a $y \in \bar{\mathfrak{u}}$ such that $\psi(y) \not\sim \psi(x)$. We apply Proposition 16 to $\Pi = \Pi(1, x, y)$ and obtain a contradiction as in the proof of Lemma 27.

Let $\mathfrak{l}(1,x)$ be of type $(\bar{\mathrm{N}})$. We claim that Π does not contain lines of type (F). To exclude this possibility, let $\mathfrak{l} = \mathfrak{l}(z,t) \in \Pi$ be such a line with a generic ψ-value equal to $s \in A$.

Take points $x_1, x_2 \in \mathfrak{l}(1,x)$ such that $\psi(x_1) \neq \psi(x_2)$. This is possible since ψ takes at least two values on $\mathfrak{l}(1,x)$. Choose $y_1, y_2 \in \mathfrak{l}(1,y)$ and $\psi(y_1) \neq \psi(y_2)$ and are both not equal to $1 \in A$, and this is possible because ψ takes at least three values on $\mathfrak{l}(1,y)$ which is of type $(\bar{\mathrm{I}})$.



Moreover, we can assume that the lines $\mathfrak{l}_{ij} := \mathfrak{l}(x_i, y_j)$ do not pass through the distinguished point $\mathfrak{q} \in \mathfrak{l}(z,t)$ (where ψ takes the non-generic value). Thus $\mathfrak{l}_{ij} := \mathfrak{l}_{ij} \cap \mathfrak{l}(z,t)$ is a generic point of $\mathfrak{l}(z,t)$, which differs from x_1, x_2, y_1, y_2. Then

$$\frac{\psi(x_i)}{s} \sim \frac{\psi(y_1)}{s} \sim \frac{\psi(y_2)}{s},$$

for both $i = 1, 2$. Hence

$$1 \neq \frac{\psi(x_1)}{\psi(x_2)} \sim \frac{\psi(y_1)}{\psi(y_2)} \neq 1.$$

Therefore, $\psi(x) \sim \psi(y)$, a contradiction.

Thus, for every $\mathfrak{l} \subset \Pi(1,x,y)$ the restriction $\psi_{\mathfrak{l}}$ is induced from a map $\bar{\psi}_{\mathfrak{l}} : \mathbb{P}^1(\mathbb{Z}/p) \to A$. Now we apply Lemma 21. In Case (a) of that Lemma, the exceptional line is $\mathfrak{l}(1,y)$ and of type $(\bar{\mathrm{I}})$ and hence the restriction of ψ to any other line is either constant or of type $(\bar{\mathrm{F}})$, a contradiction the assumption that $\mathfrak{l}(1,x)$ is of type $(\bar{\mathrm{N}})$. Cases (b) and (c) are excluded: ψ is not induced from an injective map, nor a flag map on $\mathfrak{l}(1,x)$. $\square$

Lemma 30. *Assume that the pair of lines* $(\mathfrak{l}(1,x), \mathfrak{l}(1,y))$ *is of one of the following types*

$$(\mathrm{N},\mathrm{N}), \quad (\mathrm{N},\bar{\mathrm{N}}), (\mathrm{N},\bar{\mathrm{F}}), \quad (\bar{\mathrm{N}},\bar{\mathrm{N}}).$$

Then $\psi(x) \sim \psi(y)$.

Proof. Follows from the same arguments as in Lemma 29 and Lemma 27. $\square$

Lemma 31. *Assume that $\psi(\bar{\mathfrak{u}})$ contains algebraically independent elements. Consider $\mathfrak{l} := \mathfrak{l}(1,z) \not\subseteq \bar{\mathfrak{u}}$, and assume that $\mathfrak{l} \cap \bar{\mathfrak{U}}$ consists of at least two points $1, z'$. Then $\psi(z') = 1$ and $\mathfrak{l} \cap \bar{\mathfrak{U}}$ is either*

(1) $\mathfrak{l}$;
(2) *an affine line, with ψ not constant on $\mathfrak{l}$;*
(3) *projectively equivalent to $\mathbb{Z}_{(p)} \subset \mathbb{P}^1(\mathbb{Q})$;*
(4) *an affine line and ψ is constant on $\mathfrak{l}$.*

Proof. Assume that $\mathfrak{l} \notin$ (C). Write $z' = x/x'$ with $x, x' \in \bar{\mathfrak{u}}$.

- If $\psi(x) \neq \psi(x')$ then $x/x' \in \bar{\mathfrak{u}}$, by Equation 14, thus $\mathfrak{l} \subset \bar{\mathfrak{u}} \subset \bar{\mathfrak{U}}$, contradiction, so that $\psi(z') = 1$.
- If $\psi(x) = \psi(x') \neq 1$, choose a point $t \in \bar{\mathfrak{u}}$ be such that $\psi(t) \not\sim \psi(x')$, and such that it is also algebraically independent from a nontrivial value on $\mathfrak{l}$. Then $t/x' \in \bar{\mathfrak{u}}$ and the restriction of ψ to (a shift of) $\mathfrak{l}(t, x/x')$ is of type ($\bar{\text{I}}$). In particular, $\mathfrak{l}(1,t), \mathfrak{l}(t, x/x')$ are also of type ($\bar{\text{I}}$), by the same argument as in the proof of Lemma 29.

This lemma implies that $\mathfrak{l}$ is of type (F) or ($\bar{\text{F}}$).

- $\mathfrak{l} \in$ (F). In the notation of Proposition 16, ψ is of type b) on $\Pi(1,t,z)$ and the restriction of ψ to every line in $\Pi(1,t,z)$, not passing through a distinguished point $\mathfrak{q} \in \mathfrak{l}$, with $\psi(\mathfrak{q}) \neq 1$, is of type ($\bar{\text{I}}$), which implies that $\mathfrak{l} \setminus \mathfrak{q} \subset \mathfrak{U}$, i.e., we are in Case (V), i.e., the valuation case.
- $\mathfrak{l} \in$ ($\bar{\text{F}}$). In this case, $\Pi(1,t,z)$ does not contain lines of type (F), because otherwise, by Proposition 16, $\mathfrak{l}$ will also be of type (F). Hence ψ is induced from $\mathbb{P}^1(\mathbb{Z}/p)$ on any line in $\Pi(1,t,z)$ and there are two independent values of ψ on $\Pi(1,t,z)$ not equal to 1. Then ψ on $\Pi(1,t,z)$ is induced from $\bar{\psi} : \mathbb{P}^2(\mathbb{Z}/p) \to A$, by Lemma 21.

  The map $\bar{\psi}$ is injective on $\mathfrak{l}(1,t')$ and $\mathfrak{l}(t', z_1')$, where both t', z_1' are the images of t, z' under the reduction map, and a flag map on $\mathfrak{l}(1, z_1)$, where z_1 is the image of z in $\mathbb{P}^2(\mathbb{Z}/p)$. Thus ψ is induced from type b), and hence $\mathfrak{U} \cap \mathfrak{l}$ consists of y, with $\psi(y) = 1$, a set projectively equivalent to $\mathbb{Z}_{(p)} \subset \mathbb{P}^1(\mathbb{Q})$, and we are in Case (P), the projection case.

Assume that $\mathfrak{l} \in$ (C). Here the difficulty is that $\psi(\Pi(1,z,t))$ does not contain algebraically independent elements and we cannot apply Lemma 21. Note that $\mathfrak{l}(t,s)$, for $s = r/r', r, r' \in \mathfrak{u}$, $s \in \mathfrak{l}$, are of type ($\bar{\text{I}}$), by the argument above.

Then any line $\mathfrak{l}(t', u') \subset \Pi(1, z, t)$, with $\psi(t') \neq \psi(u')$, is of type $(\bar{\mathrm{I}})$, since ψ takes at least three values on this line. Hence $s := \mathfrak{l}(t', u') \cap \mathfrak{l} \in \bar{\mathfrak{U}}$.

On the other hand, if $s' \in \mathfrak{l}$ is not in $\bar{\mathfrak{U}}$, then there are at most two values on any line containing s', including $\psi(s') = 1$. We split all points into subsets:

(1) $S_T := \{x \,|\, \psi(x) \neq 1\}$;
(2) $S_1 := \{x \in \bar{\mathfrak{U}} \,|\, \psi(x) = 1\}$;
(3) $S_2 := \{x \notin \bar{\mathfrak{U}} \,|\, \psi(x) = 1\}$.

Note that $S_T, S_1 \neq \emptyset$. If $S_2 = \emptyset$ then $\mathfrak{l} \subset \bar{\mathfrak{U}}$; and $\bar{\mathfrak{U}} \cap \mathfrak{l}$ satisfies the lemma.

Assume that $S_2 \cap \mathfrak{l} \neq \emptyset$. We claim that every line in $\Pi(1, z, t)$ lies in the union of two of such subsets. Clearly, this holds for $\mathfrak{l}$. Let $\mathfrak{l}' \subset \Pi(1, z, t)$ be a different line and put $s := \mathfrak{l} \cap \mathfrak{l}'$. If $s \in \bar{\mathfrak{U}}$, then $\mathfrak{l}(s, t) \subset \bar{\mathfrak{U}}$, by construction, and all points $s \in \mathfrak{l} \cap S_T$ are in $\bar{\mathfrak{u}}$ and those with $\psi(s) = 1$ in $\bar{\mathfrak{U}}$. In particular, $\mathfrak{l}(s, t) \subset S_T \sqcup S_1$. If $s' \in \mathfrak{l}$ is in S_2, then $\mathfrak{l}(s', x)$ is of type (F), $(\bar{\mathrm{F}})$ or (C), and hence ψ takes at most two values on $\mathfrak{l}(s', x)$, including $\psi(s') = 1$.

If $s_2 \in \mathfrak{l}(s', x), \psi(s_2) = 1, x \in \bar{\mathfrak{u}}, \psi(x) \neq 1$, then $s_2 \in S_2$. Otherwise, if $s_2 \in \mathfrak{U}, x \in \bar{\mathfrak{u}}$, and then ψ is injective on $\mathfrak{l}(s', x) = \mathfrak{l}(s_2, x)$, by the argument above. Hence $s_2 \in S_2$. Thus $\mathfrak{l}(s_2, t')$, with $t' \in \mathfrak{l}(s, t)$, is contained either in $S_2 \sqcup S_T$ or $S_1 \sqcup S_2$.

Any $y \in \Pi(1, z, t)$, with $\psi(y) \neq 1$, is contained in $\bar{\mathfrak{u}}$. Indeed, consider $\mathfrak{l}(y, y')$, with $\psi(y) \neq \psi(y'), y' \in \mathfrak{l}(t, s), \psi(y') \neq 1$, and $s_y := \mathfrak{l}(y, y') \cap \mathfrak{l}(1, z)$. Then $\psi(s_y) = 1$, hence $y'/s_y \in \bar{\mathfrak{u}}$, and ψ is injective on $\mathfrak{l}(y, y')$. Since $y' \in \bar{\mathfrak{u}}$, we find that $y \in \bar{\mathfrak{u}}$ and $s_y \in S_1$.

Thus $S_T \subset \bar{\mathfrak{u}}$ and any line $\mathfrak{l}(y, s)$, with $\psi(s) = 1$, is either contained in $S_2 \sqcup S_T$ or in $S_T \sqcup S_1$. This implies that any $\mathfrak{l}(s, s_2)$, with $s \in S_1, s_2 \in S_2$, is contained in $S_1 \sqcup S_2$. Note that none of the lines is contained in one of the subsets S_T, S_1, S_2. By Proposition 5, the decomposition $\Pi = S_T \sqcup S_1 \sqcup S_2$ is either

(1) a cone over the decomposition of $\mathfrak{l}(t, s)$ into the intersection with S_T and S_1, and S_2 is just one point in $\mathfrak{l}$;
(2) or is induced from a decomposition of $\mathbb{P}^2(\mathbb{Z}/p)$ over the residue of $\mathfrak{l}$, with S_1 equal to the preimage of a point, and hence $S_2 \cap \mathfrak{l}$ is projectively equivalent to $\mathbb{Z}_{(p)}$.

□

7. Proof of the main theorem

We turn to the proof of Theorem 2, describing the homomorphisms

$$\psi : \mathbb{P}(K) \to \mathbb{P}(L),$$

preserving algebraic dependence. There are two possibilities:

(V) ψ factors through a valuation,
(P) ψ factors through a subfield,

described in detail in the Introduction.

We organize our proof as a case by case analysis, based on types of line, introduced at the beginning of Section 6. We consider two sets of cases as follows.

- Generic cases: $\psi(\mathfrak{u})$ (respectively, $\psi(\bar{\mathfrak{u}})$), contains nonconstant algebraically independent elements, i.e., there exist $y_1, y_2 \in \psi(\mathfrak{u})$ (respectively, $\psi(\bar{\mathfrak{u}})$) such that $y_1 \not\sim y_2$.
- Degenerate cases: these sets do not contain algebraically independent elements.

In our proof we need the following technical assumption:

(AD) $\psi(\bar{\mathfrak{u}})$ does not contain nonconstant algebraically independent elements.

This is satisfied when K has positive characteristic. In characteristic zero, this assumption allows us to avoid the case of geometric valuations which are induced from fields of positive characteristic.

Lemma 32. *Assume that $\psi(\mathfrak{u})$ contains nonconstant algebraically dependent elements and that $\mathbb{P}(K)$ contains lines of type* (F) *and possibly also* (C). *Then there exists a valuation ν of K such that $\mathfrak{o}_\nu^\times \subseteq \mathfrak{U}$ and $\psi((1+\mathfrak{m}_\nu)^\times) = 1$.*

Proof. By Lemma 24, $\mathfrak{U} \subset K^\times/k^\times$ is a group, the induced quotient map $K^\times/k^\times \to K^\times/\mathfrak{U}$ is a nontrivial flag map, by the assumption on the existence of lines of type (F) in $\mathbb{P}(K)$ and by Proposition 5 und using Theorem 6 and Lemmas 7, 9. By Proposition 5, there is a map

$$\mathfrak{o}_\mu^\times \to K^\times \to \Gamma_\mu,$$

for some valuation μ, with the property that $K^\times \to K^\times/\mathfrak{U}$ is a composition

$$K^\times \to \Gamma_\mu \xrightarrow{r_\mu} K^\times/\mathfrak{U}.$$

Let

$$\Gamma_\mu^+ := \nu(\mathfrak{o}_\mu \setminus 0) \subset \Gamma_\mu$$

be the subsemigroup of positive elements and put

$$\mathrm{Ker}(r_\mu)^+ := \mathrm{Ker}(r_\mu) \cap \Gamma_\mu^+.$$

- Assume that $\mathrm{Ker}(r_\mu)^+ = 0$. Then for any nonconstant

  $$x \in \mathfrak{o}_\mu^\times/(k^\times \cap \mathfrak{o}_\mu^\times) \subset \mathfrak{u}, \quad y \in (\mathfrak{m}_\mu \setminus 0)/(k^\times \cap \mathfrak{o}_\mu^\times),$$

  the restriction of ψ to $\mathfrak{l}(x, y)$ is a flag map with generic value 1 by Proposition 16, Case (c), with $y = \mathfrak{q}$. Indeed, the assumption implies that the generic value on the line $\mathfrak{l}(1, y/x)$ is 1, since $\nu(y/x) > 0$. Since $\nu(x) = 0$, the same holds for the line $\mathfrak{l}(x, y)$. Hence the result holds for $\nu = \mu$.
- Assume that $\mathrm{Ker}(r_\mu)^+ \neq 0$. Assume in addition that there exists a $\gamma^+ \in (\Gamma_\mu^+ \setminus \mathrm{Ker}(r_\mu)^+)$ and such that $\gamma^+ < \gamma'$ for some $\gamma' \in \mathrm{Ker}(r_\mu)^+$. Consider $x \in (\mathfrak{u} \setminus 1)$, with $\mu(x) = \gamma'$, and $y \in \mathfrak{o}_\mu^\times/(k^\times \cap \mathfrak{o}_\mu^\times)$, with $\mu(y) = \gamma^+$. The restriction of ψ to $\mathfrak{l}(1, y) \subset \mathbb{P}(1, x, y)$ is a flag map with generic value 1, for the same reason as avove. On the one hand, $\mathfrak{l} := \mathfrak{l}(x, y) \not\subset \mathfrak{u}$, hence $\psi_\mathfrak{l}$ is a flag map, with generic value $\psi(x)$. On the other hand, the generic value of ψ on $\mathfrak{l}(1, y)$ is 1, hence $\psi(x + y) = \psi(x)$ and $x + y \in \mathfrak{u}$. We have $\mu(y) < \mu(x)$ and, on $\mathfrak{l}(x, y)$, we have $\mu(x + y) = \mu(y)$, hence $\psi(x + y) = \psi(y)$, a contradiction.

  This implies that the elements of $\mathrm{Ker}(r_\mu)^+$ are smaller than all elements in $(\Gamma_\mu^+ \setminus \mathrm{Ker}(r_\mu)^+)$. Thus the subgroup of Γ_μ generated by $\mathrm{Ker}(r_\mu)^+$ is an ordered subgroup. The homomorphism $\Gamma_\mu \to \Gamma_\mu/\mathrm{Ker}(r_\mu)^+$ identifies $\Gamma_\mu/\mathrm{Ker}(r_\mu)^+$ with a valuation group Γ_ν for some valuation ν of K, and $\psi((1+\mathfrak{m}_\nu)^\times) = 1$.

□

We can also treat all degenerate cases, i.e., $\psi(\mathfrak{u})$ and $\psi(\bar{\mathfrak{u}})$ do not contain nonconstant algebraically independent elements.

Most degenerate case: no (I)*,* ($\bar{\mathrm{I}}$)*,* (N)*, and* ($\bar{\mathrm{N}}$)*-lines*:

- Then ψ is a flag map on all $\mathfrak{l} \subset \mathbb{P}(K)$, hence a flag map, by Proposition 5 and Lemma 7, note that A has no 2-torsion. Thus there exists a valuation ν such that ψ factors through Γ_ν, and we are in Case (V) of Theorem 2, since $\psi(\mathfrak{o}_\nu^\times) = 1$.

Degenerate case: no (I) *and* ($\bar{\mathrm{I}}$)*-lines, but* (N) *or* ($\bar{\mathrm{N}}$)*-lines*:

- If there exist (N) or ($\bar{\mathrm{N}}$)-lines then by Lemma 30, there exists a 1-dimensional subfield $L_1 \subset L$ such that the images of all such lines are contained in $L_1^\times/l^\times$. Consider the induced projection homomorphism

$$\psi_1 : \mathbb{P}(K) \to L^\times/l^\times \to L^\times/L_1^\times.$$

Note that the restriction of ψ_1 to any line $\mathfrak{l} \in \mathbb{P}(K)$ is a flag map, and there exist lines on which it is a nontrivial flag map, since the image of ψ contains at least two algebraically independent elements. Thus there is a nontrivial valuation μ of K such that ψ_1 factors through the value group Γ_μ.

Degenerate case: there exist (I)*-lines* $\mathfrak{l}$ *and* $\psi(\mathfrak{l}) \subset L_1^\times/l^\times$*, for some 1-dimensional field* $L_1 \subset L$.

- Let L_2 be the algebraic closure of L_1 in L. There may also exist lines $\mathfrak{l} \subset \mathbb{P}(K)$ of type (N), ($\bar{\text{N}}$), ($\bar{\text{I}}$), or ($\bar{\text{F}}$), with respect to ψ, but $\psi(\mathfrak{l}) \subset L_2^\times/l^\times$ for all such $\mathfrak{l}$, by Lemma 27. Again, every $\mathfrak{l} \subset \mathbb{P}(K)$ is either of type (C) or (F), with respect to

  $$\psi_2 : \mathbb{P}(K) \to L^\times/l^\times \to L^\times/L_2^\times,$$

  and there exists a nontrivial valuation μ of K such that ψ_2 factors through Γ_μ.

Degenerate case: there are no (I)*–lines but there exist* ($\bar{\text{I}}$)*-lines whose images are contained in* $L_1^\times/l^\times$*, for some 1-dimensional subfield of* L.

- Let L_2 be its algebraic closure in L. There may exist lines of type (N), ($\bar{\text{N}}$), or ($\bar{\text{I}}$), but their images are contained in $L_2^\times/l^\times$. Every $\mathfrak{l} \subset \mathbb{P}(K)$ is of type (C), (F), or ($\bar{\text{F}}$), with respect to

  $$\psi_2 : \mathbb{P}(K) \to L^\times/l^\times \to L^\times/L_2^\times,$$

  and there exists a nontrivial valuation ν of K such that ψ_2 factors through Γ_μ.

Thus, in all the degenerate cases the homomorphism

$$\psi_\ell : K^\times/k^\times \to L^\times/L_2^\times,$$

is a flag map, thus arises from a nontrivial valuation μ,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{o}_\mu^\times & \longrightarrow & K^\times & \xrightarrow{\mu} & \Gamma_\mu & \longrightarrow & 1 \\ & & & & \| & & \downarrow{\scriptstyle r} & & \\ & & & & K^\times & \xrightarrow{\psi_\ell} & L^\times/L_2^\times & & \end{array}$$

i.e., $\psi_\ell = r \circ \mu$. The following lemma will show that ψ is either as in (V) or (VP) of Theorem 2, by constructing a valuation as required, using ψ.

Lemma 33. *There is a valuation ν of K and a surjective homomorphism of ordered groups*

$$\Gamma_\mu \xrightarrow{\gamma} \Gamma_\nu$$

such that

(1) $\nu = \gamma \circ \mu : K^\times \to \Gamma_\nu$ *is a surjective map of ordered groups with* $\mathrm{Ker}(\gamma) \subset \mathrm{Ker}(r)$.
(2) $\psi((1+\mathfrak{m}_\nu)^\times) = 1$.

Proof. Let $z \in \mathfrak{o}_\mu$ be such that $r(\mu(z)) \neq 0$ and thus $\psi_\ell(z) \neq 1 \in L^\times/L_2^\times$. Let $x \in \mathfrak{o}_\mu^\times \subset \mathrm{Ker}(\psi_\ell)$. We have

$$\mu(x+az) = \mu(x), \mu(a) \geq 0,$$

and r is nonconstant on $\mathfrak{l}(z,x)$. Thus ψ is a flag map on $\mathfrak{l}$, and

$$\psi(x+az) = \psi(x)$$

so that $\psi(1+az/x) = 1$. Note that zx also has $r(\mu(zx)) \neq 0$ and hence we can apply the same to zx, obtaining $\psi(1+az) = 1$, for any z with $r(\mu(z)) > 0$.

Note that elements z with $\mu(z) = \alpha$ generate additively the subgroup $K_\alpha \subset K$. Now the elements of the form $1+z$ with $\mu(1+z) = 0$ generate the multiplicative subgroup $(1+K_\alpha)^\times$. Indeed, consider

$$(1+z)(1+z') = 1+z+z'+zz' = (1+z+z')\left(1+\frac{zz'}{1+z+z'}\right),$$

where $\mu(z) = \mu(z')$ and $(1+z+z') \in \mathfrak{o}_\mu^\times$. Since $\psi_\ell(zz') \neq 1$ we have

$$\psi\left(1+\frac{zz'}{1+z+z'}\right) = 1,$$

by the same argument applied to z, z'; thus $\psi \equiv 1$ on $(1+K_\alpha)^\times$. This implies that $\psi(1+y) = 1$, even if $r(\mu(y)) = 0$, but there is a $z, r(\mu(z)) \neq 1$ and $\mu(z) < \mu(y)$. Consider the subset $\Gamma_\mu^+, \mu \geq 0$ in Γ_μ. Since $L^\times/L_2^\times$ is torsion-free,

$$\mathrm{rk}_{\mathbb{Q}}(\mathrm{Ker}(r)) < \mathrm{rk}_{\mathbb{Q}}(\Gamma_\mu).$$

Hence $\mathrm{Ker}(r)$ intersects Γ_μ^+ in a proper subsemigroup $\mathrm{Ker}(r_\mu)^+$ and the subset of elements $s \in \mathrm{Ker}(r_\mu)^+$ with $s > \mu(x)$ for any $x \in \Gamma_\mu^+ \setminus \mathrm{Ker}(r_\mu)^+$.

We are looking at a subset of elements S inside $\mathrm{Ker}(r_\mu)^+ \setminus 0$ such that for each $s \in S$ such that $s < u$ for any $u > 0$ with $r(u) \neq 0$. Note that S has to contain the smallest elements in $\Gamma_\mu^+ \setminus 0$ if they exist. Assume that $s, s' \in S, s, s' < u, r(u) \neq 0$ and $s + s' > u$. Note that $s + s' - u > 0$ and $s > u - s' > 0$ but $r(u-s') \neq 0$ which provides a contradiction. Thus S is an ordered subsemigroup in $\mathrm{Ker}(r_\mu)^+ \setminus 0$ which generates an ordered subgroup $\langle S \rangle$ such that

$$K^\times \to \Gamma_\mu/\langle S \rangle =: \Gamma_\nu$$

is a valuation map for some valuation ν. For this valuation, $\mathrm{Ker}(\nu) \supset (1+\mathfrak{m}_\nu)^\times$, by the computation above. □

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