

# NOETHER'S PROBLEM AND DESCENT

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ABSTRACT. We study Noether's problem from the perspective of torsors under linear algebraic groups and descent.

## INTRODUCTION

This note is inspired by the theory of universal torsors developed by Colliot-Thélène and Sansuc in connection with arithmetic problems on del Pezzo surfaces [CTS87b]. This theory associates to a geometrically rational surface  $X$  over a field  $k$ , with  $X(k) \neq \emptyset$ , a torsor

$$(1) \quad \pi : \mathcal{T} \xrightarrow{T} X,$$

where  $T$  is the Néron-Severi torus of  $X$ , i.e., the character group of  $T_{\bar{k}}$  is isomorphic, as a Galois module, to the geometric Néron-Severi lattice  $\text{NS}(X_{\bar{k}})$  of  $X$ . The torsor is viewed as a *descent* variety. Basic arithmetic problems on  $X$ , such as Weak Approximation, are reduced to the following geometric hypothesis:  $\mathcal{T}$  is *rational* over  $k$ . The gist of this hypothesis is that on  $\mathcal{T}$ , the arithmetic complexity of  $X$  is *untwisted*: while  $X$  may have nontrivial (algebraic) Brauer group, it is eliminated via passage to a universal torsor. A positive answer to this would imply in particular that  $X$  is unirational over  $k$ , which is an open problem for del Pezzo surfaces of degree 1.

In higher dimensions, unirational varieties may have nontrivial transcendental Brauer group, or more generally, nontrivial higher unramified cohomology. Examples are conic bundles over rational surfaces considered by Artin–Mumford [AM72], quadric bundles [CTO89], or Brauer–Severi bundles. Our motivation was to understand whether or not these obstructions to stable rationality can be untwisted via passage to fibrations as in (1).

**Definition 1.** A variety  $X$  over a field  $k$  admits a *rational tower* if there exists a sequence of dominant rational maps

$$(2) \quad X_n \xrightarrow{\pi_n} X_{n-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0 := X,$$

over  $k$ , such that

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- (1) the *source* of the tower,  $X_n$ , is rational over  $k$ ,
- (2) the generic fiber of  $\pi_i$  is geometrically rational and irreducible, over the function field  $k(X_{i-1})$ , for all  $i$ .

We say that  $X$  admits a *toric rational tower*, if in addition

- (2') the generic fiber of  $\pi_i$  is birational to a torsor under an algebraic torus, over the function field  $k(X_{i-1})$ , for all  $i$ .

**Conjecture 2.** *Let  $X$  be a unirational variety over  $k$ . Then  $X$  admits a rational tower.*

This conjecture is motivated by topology: every continuous map is homotopy equivalent to a fibration with constant fiber. In the algebraic-geometric context, the conjecture implies in particular that *unramified cohomology* of the source of the tower,  $X_n$ , in Definition 1, is trivial. Of course, one can trivialize a given unramified cohomology class (with finite coefficients) via passage to a tower with geometrically rational fibers, as in (2). Indeed, let  $K = k(X)$  be the function field of an algebraic variety over a field  $k$ , which contains all roots of unity. By the Bloch-Kato conjecture, proved by Voevodsky, the subring

$$\bigoplus_{i \geq 2} H^i(K) \subset H^*(K)$$

of Galois cohomology of  $K$ , with finite coefficients, equals the ideal generated by  $H^2(K)$ . Since elements of  $H^2$  are trivialized on the corresponding Brauer-Severi variety, we can trivialize an arbitrary finite set of Galois cohomology classes via passage to a tower as in (2):

$$Y_m \rightarrow Y_{m-1} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = X,$$

where the generic fibers are forms of projective spaces. However, a direct application of this construction does not imply Conjecture 2, as we cannot assert the rationality of  $Y_m$ . In fact, new unramified classes may appear in the process, and we cannot even ensure that  $Y_m$  has trivial unramified cohomology.

To motivate Conjecture 2 from the perspective of algebra, consider the following important class of unirational varieties: quotients  $V/G$ , where  $G$  is a finite group and  $V$  a (finite-dimensional) faithful linear representation of  $G$  over  $k$ . In [BT17] we showed that, over  $k = \overline{\mathbb{F}}_p$ , these varieties are *universal* for unramified cohomology: every unramified cohomology class of an algebraic variety over  $k$  is induced from  $V/G^c$ , where  $G^c$  is a central extension of an abelian group (see Section 1 for more details). Thus we expect that such quotients are universal from the birational point of view as well. For groups of this type, and a wide class of solvable groups  $G$ , we prove:

- $V/G$  admits a toric rational tower.

Its source  $X_n$  is a  $k$ -rational variety, untwisting all unramified cohomology of  $V/G$ .

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## 1. BACKGROUND

Let  $X$  be a smooth projective geometrically rational variety over a field  $k$ . Among obstructions to  $k$ -rationality are the absence of  $k$ -rational points or the nontriviality of the Brauer group

$$\mathrm{Br}(X)/\mathrm{Br}(k).$$

More generally, we may consider *unramified cohomology*

$$H_{nr}^i(X, \mu_m^{\otimes i-1}), \quad H_{nr}^i(X, \mathbb{Q}/\mathbb{Z}(i-1)), \quad m \in \mathbb{N},$$

(see [Bog87a], [CTO89] for definitions and basic properties). These groups are birational invariants which vanish when  $i > \dim(X)$  or when  $X$  is  $k$ -rational, when  $k$  is algebraically closed. For  $X$  smooth and projective, we have

$$\mathrm{Br}(X)[m] = H_{nr}^2(X, \mu_m),$$

where  $\mu_m$  stands for  $m$ -th roots of 1. Below, we omit the coefficients, when they are clear from the context.

**Conjecture 3.** *Let  $X$  be a unirational variety over an algebraically closed field of characteristic zero. Then its unramified cohomology is finite.*

Finiteness of  $H_{nr}^i(X, \mathbb{Q}/\mathbb{Z}(i-1))$ , for unirational  $X$ , is known for:

- $i = 2$ , classically,
- $i = 3$  [CTK13, Proposition 3.2].

When  $X(k) \neq \emptyset$ , the theory of Colliot-Thélène–Sansuc provides a fibration

$$\pi : \mathcal{T} \xrightarrow{T} X$$

as in (1), such that

- the generic fiber of  $\pi$  is a principal homogeneous space under an algebraic torus  $T$  and

- if  $\mathcal{T}(k) \neq \emptyset$  then  $\mathcal{T}$  is, *hypothetically*, a rational variety [CTS87b, Hypothesis H1, Section 2.8].

Special cases of this are known, e.g., when  $X$  admits a conic bundle over  $k$ , with 4 degenerate fibers [CTS87a]. As further evidence for the rationality hypothesis, one has the proof that, over  $k$  of characteristic zero,  $H_{nr}^2(\mathcal{T})$  is trivial [CTS87b, Theorem 2.1.2]. A more recent consistency check is [Cao16a] and [Cao16b], where it is shown that

- when  $X$  is a geometrically rational surface, the nontrivial part of  $H_{nr}^3(\mathcal{T})$  is finite and 2-primary.

An example of a geometrically rational variety of dimension 3 where Hypothesis H1 fails is given in [CTS87b, Remarque 2.8.18].

We now turn to unirational varieties which are not necessarily geometrically rational, which may have nontrivial *transcendental* Brauer group. Examples appeared in the context of *Noether's problem*:

**Problem 4** (Noether). *Let  $V$  be a faithful representation of a finite group  $G$  over  $k$ . Is  $V/G$  (stably) rational?*

Noether's problem has a negative solution: counterexamples are based on explicit computations of unramified cohomology, which is essentially a combinatorial problem, in terms of the structure of Sylow subgroups of  $G$ . Groups  $G$  with nontrivial  $H_{nr}^2(V/G)$  were constructed in [Sal84], [Bog87b], and [Mor12]; with trivial  $H_{nr}^2(V/G)$  but nontrivial  $H_{nr}^3(V/G)$  in [Pey08] and [HKY16].

On the other hand, unramified cohomology is trivial for actions of many classical finite groups. In some, but not all, of these cases, we have proofs of (stable) rationality (see, e.g., [BK85], [BBGVB11]).

Our goal in the following sections will be to *untwist* unramified cohomology of  $V/G$  by constructing rational towers as in Definition 1. Of particular importance are solvable groups  $G$ . Indeed, in [BT17] we proved a *universality* result:

**Theorem 5.** *Let  $X$  be an algebraic variety of dimension  $\geq 2$  over  $k = \bar{\mathbb{F}}_p$ ,  $\ell \neq p$  a prime, and*

$$\alpha \in H_{nr}^i(X, \mathbb{Z}/\ell^n)$$

*an unramified class. Then there exist finite-dimensional  $k$ -vector spaces  $V_j, j \in J$ , depending on  $\alpha$ , such that  $\alpha$  is induced, via a rational map, from an unramified class in the cohomology of the quotient of*

$$\mathbb{P} := \prod_{j \in J} \mathbb{P}(V_j)$$

*by a finite abelian  $\ell$ -group  $G^a$ , acting projectively on each factor.*

In other words, central extensions of abelian groups capture all unramified cohomology invariants.

## 2. FIRST PROPERTIES

Throughout,  $G$  is a linear algebraic group over a field  $k$  and  $V$  a finite-dimensional linear faithful representation of  $G$  over  $k$ .

Let  $\text{Bir}(k)$  be the set algebraic varieties over  $k$ , modulo  $k$ -birational equivalence, which we denote by  $\sim_k$ . Let

$$\text{Rat}(k) \subset \mathcal{LQ}(k) \subset \mathcal{GQ}(k) \subset \text{Unirat}(k),$$

be the classes of algebraic varieties over  $k$  which are

- $k$ -rational,
- $k$ -birational to  $V/G$  (Linear Quotients),
- $k$ -birational to  $X/G$ , where  $X$  is a  $k$ -rational algebraic variety and  $G$  is a subgroup of the group  $\text{BirAut}(X)$  of  $k$ -birational automorphisms of  $X$  (General Quotients),
- $k$ -unirational,

respectively. Our goal is to connect these classes, via passage to fibrations with geometrically rational generic fibers as in Definition 1. We start with simple examples:

*Example 6.* Let  $G$  be a finite abelian group, or an extension of a finite cyclic group by a finite abelian group. If  $k$  is of characteristic coprime to  $n := |G|$  and contains  $n$ -th roots of 1 then  $V/G$  is rational (see [Swa83, Theorem 6.1]).

This can fail, in general, even when  $G$  is cyclic. Over  $\mathbb{Q}$ , there are counterexamples, e.g., for  $G = \mathbb{Z}/8\mathbb{Z}$  [Sal82, Theorem 5.11], or for  $G = \mathbb{Z}/47\mathbb{Z}$  [Swa69], [Len74], [Pla17].

*Example 7.* Let  $V$  be a faithful linear representation of a finite group  $G$  over  $k$ , i.e.,  $G \hookrightarrow \text{GL}(V)$ . Then

$$V/G \sim_k \mathbb{P}(V)/G \times \mathbb{P}^1.$$

*Example 8.* Let  $G$  be a *connected* linear algebraic group and  $V$  a faithful linear representation of  $G$  over  $k$ . Then  $Y := V/G$  admits a rational tower. Indeed, the total space of the corresponding (rational) fibration

$$V \rightarrow V/G = Y$$

is clearly  $k$ -rational, and the generic fiber is geometrically rational.

*Example 9.* Toric varieties: a universal torsor of a toric variety  $X_\Sigma$  over  $k$  is given by

$$\mathcal{T}_\Sigma = \mathbb{A}^n \setminus Z_\Sigma,$$

where  $Z_\Sigma$  is a locally closed subvariety; we have

$$X_\Sigma = \mathcal{T}_\Sigma // T_{NS},$$

where  $T_{NS}$  is the Néron-Severi torus of  $X$ . Thus  $X_\Sigma$  admits a toric rational tower.

**Lemma 10.** *Let  $G$  be a finite group and  $Y = V/G \in \mathcal{LQ}(k)$ . Assume that there exists an  $X \in \text{Rat}(k)$  with a generically free  $G$ -action, such that  $X/G \in \text{Rat}(k)$ . Then there exists a fibration  $Y_1 \rightarrow Y$  with geometrically rational generic fiber and  $k$ -rational  $Y_1$ .*

*Proof.* We have a fibration

$$(X \times V)/G \rightarrow V/G = Y,$$

with generic fiber geometrically isomorphic to  $X$ . On the other hand, we have a *vector bundle*

$$(X \times V)/G \rightarrow X/G$$

with fiber  $V$ , since the  $G$ -action on  $V$  is linear. Thus  $(X \times V)/G$  is  $k$ -rational.  $\square$

**Corollary 11.** *Let  $X$  be a rational surface over an algebraically closed field  $k$  and  $G$  a linear algebraic group contained in  $\text{BirAut}(X)$ . Then  $Y = V/G$  admits a rational tower.*

*Remark 12.* Let  $C \rightarrow \mathbb{P}^1$  be a Galois cover with Galois group  $G$ . Then there exists a fibration

$$\pi : (C \times V)/G \rightarrow V/G$$

with rational total space. However, the generic fiber of  $\pi$  is not necessarily geometrically rational; it is rational iff the genus  $g(C) = 0$ .

### 3. CENTRAL EXTENSIONS AND WREATH PRODUCTS

In this section, we investigate the existence of rational towers for  $\mathcal{LQ}(k)$ , over algebraically closed fields  $k$  of characteristic zero. We prove Conjecture 2 for  $V/G$ , for special groups  $G$ .

It is well-known that for finite abelian groups  $A$  and their faithful linear representations  $V$ , the quotient  $V/A$  is rational, provided the ground field  $k$  contains  $n$ -th roots of 1, where  $n = |A|$ . We turn to central extensions of abelian groups: among these, we have the free central extension

$$1 \rightarrow Z \rightarrow F^c(A) \rightarrow A \rightarrow 1,$$

with

$$Z := \wedge^2(A),$$

generated by commutators of lifts of elements of  $A$ , without nontrivial relations. The extension  $F^c(A)$  unique, modulo isoclinism.

**Lemma 13.** *Consider a central extension of an abelian group*

$$1 \rightarrow Z \rightarrow G^c \rightarrow A \rightarrow 1.$$

*Then there exists an exact sequence*

$$(3) \quad 1 \rightarrow \tilde{Z} \rightarrow \tilde{F}^c \rightarrow G^c \rightarrow 1,$$

*where  $\tilde{F}^c$  is isoclinic to  $F^c(A)$ , and  $\tilde{Z}$  is abelian.*

*Proof.* It suffices to add to  $G^c$  additional elements which kill the image of stable cohomology  $H_{st}^2(A, \mathbb{Z}/\ell)$  in  $H^2(\tilde{F}^c, \mathbb{Z}/\ell)$  (see [BPT10, Section 4] and [BT17, Section 2] for definitions and properties).  $\square$

**Corollary 14.** *There exist a faithful representation  $W$  of  $F^c = F^c(A)$  and a fibration*

$$W/F^c \rightarrow V/G^c$$

*whose generic fiber is birational to an algebraic torus over the function field of the base.*

*Proof.* Let  $W'$  be a faithful representation of  $\tilde{F}^c$  and put  $W := W' \oplus V$ . We have a natural surjective homomorphism  $W \rightarrow V$  giving rise to a fibration

$$W/\tilde{F}^c \rightarrow V/G^c.$$

Its generic fiber is geometrically isomorphic to  $W'/\tilde{Z}$ , with  $\tilde{Z}$  defined in (3). It suffices to recall that linear quotients of abelian groups are rational, when  $k$  contains roots of unity.  $\square$

**Conjecture 15.** *Let  $A$  be a finite abelian group,  $F^c(A)$  its free central extension, and  $V$  a faithful linear representation of  $F^c$ , over an algebraically closed field  $k$ . Then  $V/F^c$  is stably rational.*

Note that stable rationality does not change within a fixed isoclinism class, over fields containing roots of unity.

*Example 16.* Conjecture 15 holds when  $A$  is cyclic. When  $A \simeq \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell$ ,  $F^c(A)$  is the Heisenberg group. The problem reduces to a *monomial* action, and Conjecture 15 holds as well. Next, consider:

$$A := \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell, \quad \text{for } \ell = 2.$$

Modulo isoclinism, we can represent

$$F^c = F^c(A) \subset Q_{8,1} \times Q_{8,2} \times Q_{8,3},$$

where each  $Q_{8,i}$  is the group of quaternions over  $\mathbb{Z}/2$ : Choose a  $\mathbb{Z}/2$ -basis  $\{e_1, e_2, e_3\}$  of  $A$  and a basis  $\pi_1, \pi_2, \pi_3$  of surjective homomorphisms

$$\pi_i : A \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

each trivial on one of the generators  $e_1, e_2$ , or  $e_3$ . This induces diagrams

$$\begin{array}{ccccccc} 1 & \longrightarrow & (\mathbb{Z}/2)^3 & \longrightarrow & F^c & \longrightarrow & (\mathbb{Z}/2)^3 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & (\mathbb{Z}/2) & \longrightarrow & Q_{8,i} & \longrightarrow & (\mathbb{Z}/2)^2 \longrightarrow 1 \end{array}$$

Let  $V_i$  be the standard 2-dimensional representation of  $Q_{8,i}$ . Then

$$W := V_1 \oplus V_2 \oplus V_3$$

is a faithful representation of  $F^c$ . We have a  $k^\times$ -action on each component that commutes with the action of  $F^c$ . It follows that

$$W/F^c \sim_k ((\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/G) \times (k^\times)^3,$$

where  $G = (\mathbb{Z}/2)^3$  acts monomially on  $k(x_1, x_2, x_3)$ , as follows:

$$(4) \quad \sigma(x_i) = c_{i,\sigma} x_i^{a_{i,\sigma}}, \quad c_{i,\sigma}, a_{i,\sigma} \in \{\pm 1\}, \quad \forall \sigma \in G.$$

By [Yam12, Theorem 10], Type (3,3,3,1) in the notation therein, the field of invariants  $k(x_1, x_2, x_3)^G$  is rational over  $k$ , when  $k$  is algebraically closed. It follows that  $W/F^c$  is stably rational.

**Lemma 17.** [BP11, Lemma 2.4] *Let  $G, H$  be finite groups, acting faithfully on  $X$  and  $Y$ , respectively. Assume that  $X/G, Y/H \in \text{Rat}(k)$ . Let  $K := H \wr G$  be the wreath product, with its natural action on  $W := X^{|H|} \times Y$ . Then  $W/K$  is rational.*

*Proof.* Observe that the quotient  $X^{|H|}/H^{|G|}$  is a product of rational varieties  $(X/G)^{|H|}$ . The group  $H$  acts on  $(X/G)^{|H|}$  by permutations, and this action is equivalent to the linear (free permutation) action of  $H$  on  $V^{\oplus |H|}$ , where  $V \sim_k X/G$ . Thus the quotient  $W/K$  is  $k$ -birational to a vector bundle over  $Y/H$ , and hence rational.  $\square$

**Definition 18.** A finite solvable group  $G$  is called *special*, if there exists a filtration

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_r = 1$$

such that, for all  $i = 0, \dots, r-1$ , one has

- $G_i$  is a normal subgroup in  $G$ ,
- the kernel of the projection

$$p_i : G/G_{i+1} \rightarrow G/G_i$$

is abelian and



- there exists a section  $s_i : G/G_i \rightarrow G/G_{i+1}$

**Proposition 19.** *Let  $G$  be a special solvable group and  $V$  a faithful linear representation of  $G$ . Then there exists a toric rational tower*

$$X_n \rightarrow \cdots \rightarrow X_0 := V/G.$$

*Proof.* By induction. We assume that the claim holds for  $G_{(i)} := G/G_i$ . The group  $G_{(i+1)}$  is uniquely defined by the action of  $G_{(i)}$  on the abelian group  $A_i := G_i/G_{i+1}$ . Then there exists a surjection

$$(5) \quad \beta_i : B_i := \mathbb{Z}/m_i[G_i]^{r_i} \rightarrow A_i,$$

for some  $r_i$  and  $m_i$ . Its kernel  $\text{Ker}(\beta_i)$  is abelian.

This induces a surjection from the wreath product  $G_{(i)} \wr (\mathbb{Z}/m_i)^{r_i}$  onto  $G_{(i+1)}$ . Let  $W_i$  be a faithful representation of  $B_i$ . Then there is a  $G$ -equivariant projection

$$W_i^{|G_i|} \rightarrow W,$$

onto some linear representation  $W$  of  $A_i$ , corresponding to the surjection of modules (5). Let  $V_i$  be a faithful representation of  $G_{(i)}$  and consider the surjection

$$V_i \oplus W_i^{|G_{(i)}|} \rightarrow V_i \oplus W.$$

Its kernel  $\text{Ker}_i$  is a linear space with a faithful action of the abelian group  $\text{Ker}(\beta_i)$ . The quotient  $\text{Ker}/\text{Ker}(\beta_i)$  is toric, it is the fiber of the projection

$$(6) \quad (V_i \oplus W_i^{|G_{(i)}|})/W_i \rightarrow (V_i \oplus W)/G_{(i+1)}.$$

By induction hypothesis and Lemma 17, the left side of (6) admits a tower of toric fibrations with rational source. This implies the claim for  $i + 1$ .  $\square$

*Remark 20.* Our argument is similar to [Sal82, Theorem 3.3] that was focused on the Inverse Galois Problem.

**Corollary 21.** *Assume that  $G$  is a special solvable group. Then there exists a rational  $G$ -variety  $X$  with  $X/G \in \text{Rat}(k)$ .*

*Proof.* By induction, as in the proof of Proposition 19. It suffices to observe that  $X_{i+1}$  is a quotient of a vector bundle over  $X_i$  by an abelian linear action on the generic fiber. The corresponding quotient is rational, when  $X_i$  is rational.  $\square$

Thus we obtain many groups with nontrivial (equivalent to linear) embedding into Cremona groups, with rational quotients.

**Corollary 22.** *Let  $V$  be a faithful representation of an  $\ell$ -group  $G^c$ , a finite central extension of an abelian  $\ell$ -group  $A$ . Then  $V/G^c$  admits a tower of toric fibrations with rational source.*

*Proof.* We apply induction on the  $\ell$ -rank of  $A$ . The claim is trivial when rank equals 1. Let  $A' := A \oplus \mathbb{Z}/\ell^m$ . Let  $F^c := F^c(A')$  be the free central extension of  $A'$ . We have a surjection

$$F^c(A') \twoheadrightarrow F^a(A),$$

with a canonical section and abelian kernel. Now we apply Proposition 19.  $\square$

As mentioned in Section 1, Noether's problem, i.e., the rationality of  $V/G$ , has a negative solution. However, one can consider more general, nonlinear, generically free  $G$ -actions on rational varieties. A version of Conjecture 2, and of Noether's problem, is the following

**Conjecture 23.** *Let  $G$  be a finite group. Then there exists a  $k$ -rational  $G$ -variety  $X$  with generically free  $G$ -action such that  $X/G$  is  $k$ -rational.*

#### 4. GROUP THEORY

In this section we consider finite simple groups  $G$ . Our main observation is that the Sylow subgroups  $\text{Syl}_\ell(G)$  of most simple groups  $G$  satisfy the assumptions of Proposition 19; the corresponding  $V/\text{Syl}_\ell(G)$  admit a rational tower. Below we sketch a proof in a special case.

Let  $G = \text{PGL}_n(\mathbb{F}_q)$ , with  $q = p^m$ , for  $m \in \mathbb{N}$ .

- (1) Consider  $\text{Syl}_p(G)$ . It is conjugate to a subgroup of upper-triangular matrices  $U_n \subset G$ . There is a natural projection

$$U_r \rightarrow U_{r-1}, \quad \forall r,$$

with a section  $s_{r-1}$  and abelian kernel, as in the assumptions of Proposition 19, which induces the corresponding structure on  $\text{Syl}_p(G)$ .

- (2) Consider  $\text{Syl}_\ell(G)$ , with  $\ell \neq p$ . Every such subgroup is a subgroup of the normalizer  $N(T)$  of a (possibly nonsplit) maximal torus  $T \subset G$ . We have an extension

$$1 \rightarrow T \rightarrow N(T) \rightarrow W_T(G) \rightarrow 1.$$

Note that  $\text{Syl}_\ell(G)$  admits a projection onto the corresponding  $\text{Syl}_\ell(W_T(G))$  and the induced extension, by an abelian  $\ell$ -group, splits (for  $\ell \neq 2$ ). Since Proposition 19 holds for

$$\text{Syl}_\ell(W_T(G)) = \text{Syl}_\ell(\mathfrak{S}_{n-1})$$

(it is an iterated wreath product extension by cyclic  $\ell$ -groups), it also holds for  $\text{Syl}_\ell(G)$ .

Similar arguments apply to other finite groups of Lie type. Additional considerations are needed for some sporadic simple groups, and small primes.

## 5. ACTIONS IN SMALL DIMENSIONS

In this section, we survey related results on rationality of quotients  $X/G$ , for low-dimensional rational varieties  $X$  over (possibly nonclosed) fields  $k$  of characteristic zero.

Let  $V$  be a faithful linear representation of a finite group  $G$  over  $k$ . When  $G$  is abelian,  $V/G$  is rational; however,  $\mathbb{P}(V)/G$  need not be  $k$ -rational, even when  $\dim(V) = 4$  [AHK00, Example 2.3].

When  $\dim(V) \leq 3$ ,  $V/G$  is rational over algebraically closed fields  $k$ : indeed, by Example 7, it suffices to consider the unirational surface  $\mathbb{P}(V)/G$  which is rational. However, the situation is different over nonclosed fields.

*Dimension 2.* There is a large literature on rational  $G$ -surfaces  $X$  over nonclosed fields, i.e., actions of finite groups  $G \subset \text{Aut}(X)$  (see [Man67]). We have  $X/G \sim_k S/G$ , where  $S$  is a  $G$ -minimal del Pezzo surface or a conic bundle. If  $S$  is a conic bundle then  $G \subset \text{PGL}_2(\bar{k})$  (see [Tre16a, Theorem 1.3]). We start with several rationality results: let  $X$  be a smooth del Pezzo surface with  $X(k) \neq \emptyset$  and  $G \subset \text{Aut}(X)$ .

- If  $\deg(X) = K_X^2 \geq 5$  then  $X/G \in \text{Rat}(k)$  [Tre14, Corollary 1.4], [Tre16b, Theorem 1.1].
- If  $\deg(X) = 3$  and  $|G| \neq 3$  then  $X/G \in \text{Rat}(k)$  [Tre16b, Theorem 1.3].

*Example 24.* [Tre16b, Section 5] Let  $X \subset \mathbb{P}^3$  be a smooth cubic surface given by

$$f_3(x, y) + zt(ux + vy) + z^3 + wt^3 = 0,$$

where  $f_3$  is a form of degree 3,  $(x : y : z : t)$  are coordinates in  $\mathbb{P}^3$ , and  $u, v, w$  are parameters. Assume that the Galois group of  $f_3$  is  $\mathbb{Z}/2$ . Then  $X$  admits an action of  $G = \mathbb{Z}/3$ , and  $X/G$  is  $k$ -birational to a nonrational over  $k$ , minimal degree 4 del Pezzo surface  $S$ , admitting a conic bundle  $S \rightarrow \mathbb{P}^1$  over  $k$ . By [CTS87a],  $X/G$  admits a rational tower: a universal torsor is  $k$ -rational.

On the other hand, over nonclosed  $k$ , the set of  $k$ -birational types of quotients of conic bundles over  $\mathbb{P}^1$  may be infinite [Tre16a, Theorem 1.8]. E.g., this holds when  $G = \mathfrak{A}_5$  and

- not every element of  $k$  is a square,
- $\sqrt{-1}, \sqrt{5} \in k$ .

**Problem 25.** *Establish the existence of rational towers for Del Pezzo surfaces over nonclosed fields.*

*Dimension 3.* Over algebraically closed  $k$ , stable rationality of quotients  $\mathbb{P}^3/G$  is unknown for

- central  $\mathbb{Z}/2$ -extensions of the following groups:

$$\tilde{\mathfrak{S}}_5, \tilde{\mathfrak{A}}_6, \tilde{\mathfrak{S}}_6, \tilde{\mathfrak{A}}_7, \mathrm{SL}_2(\mathbb{F}_7),$$

- extensions of  $\mathfrak{A}_5$ , respectively,  $\mathfrak{A}_6$  by a group  $N$  of order 64.

For all other subgroups  $G \subset \mathrm{GL}_4(k)$ ,  $\mathbb{P}^3/G$  is stably rational [Pro10]. Rationality over nonclosed fields has not been addressed.

#### REFERENCES

- [AHK00] Hamza Ahmad, Mowaffaq Hajja, and Ming-chang Kang. Rationality of some projective linear actions. *J. Algebra*, 228(2):643–658, 2000.
- [AM72] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.
- [BBGV11] Fedor Bogomolov, Christian Böhning, and Hans-Christian Graf Von Bothmer. Rationality of quotients by linear actions of affine groups. *Sci. China Math.*, 54(8):1521–1532, 2011.
- [BK85] F. A. Bogomolov and P. I. Katsylo. Rationality of some quotient varieties. *Mat. Sb. (N.S.)*, 126(168)(4):584–589, 1985.
- [Bog87a] F. A. Bogomolov. The Brauer group of quotient spaces of linear representations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(3):485–516, 688, 1987.
- [Bog87b] F. A. Bogomolov. The Brauer group of quotient spaces of linear representations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(3):485–516, 688, 1987.
- [BP11] Fedor Bogomolov and Tihomir Petrov. Unramified cohomology of alternating groups. *Cent. Eur. J. Math.*, 9(5):936–948, 2011.
- [BPT10] Fedor Bogomolov, Tihomir Petrov, and Yuri Tschinkel. Unramified cohomology of finite groups of Lie type. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 55–73. Birkhäuser Boston, Inc., Boston, MA, 2010.
- [BT17] Fedor Bogomolov and Yuri Tschinkel. Universal spaces for unramified Galois cohomology. In *Brauer groups and obstruction problems*, volume 320 of *Progr. Math.*, pages 57–86. Birkhäuser/Springer, Cham, 2017.
- [Cao16a] Yang Cao. Cohomologie non ramifiée de degré 3 : variétés cellulaires et surfaces de del Pezzo de degré au moins 5, 2016. [1607.01739](https://arxiv.org/abs/1607.01739).
- [Cao16b] Yang Cao. Troisième groupe de cohomologie non ramifié des torseurs universels sur les surfaces rationnelles, 2016. [arXiv:1609.05117](https://arxiv.org/abs/1609.05117).

- [CTK13] Jean-Louis Colliot-Thélène and Bruno Kahn. Cycles de codimension 2 et  $H^3$  non ramifié pour les variétés sur les corps finis. *J. K-Theory*, 11(1):1–53, 2013.
- [CTO89] J.-L. Colliot-Thélène and M. Ojanguren. Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford. *Invent. Math.*, 97(1):141–158, 1989.
- [CTS87a] J.-L. Colliot-Thélène and A. N. Skorobogatov.  $R$ -equivalence on conic bundles of degree 4. *Duke Math. J.*, 54(2):671–677, 1987.
- [CTS87b] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.
- [HKY16] Akinari Hoshi, Ming-chang Kang, and Aiichi Yamasaki. Degree three unramified cohomology groups. *J. Algebra*, 458:120–133, 2016.
- [Len74] H. W. Lenstra, Jr. Rational functions invariant under a finite abelian group. *Invent. Math.*, 25:299–325, 1974.
- [Man67] Ju. I. Manin. Rational surfaces over perfect fields. II. *Mat. Sb. (N.S.)*, 72 (114):161–192, 1967.
- [Mor12] Primož Morávec. Groups of order  $p^5$  and their unramified Brauer groups. *J. Algebra*, 372:420–427, 2012.
- [Pey08] Emmanuel Peyre. Unramified cohomology of degree 3 and Noether's problem. *Invent. Math.*, 171(1):191–225, 2008.
- [Pla17] Bernat Plans. On Noether's rationality problem for cyclic groups over  $\mathbb{Q}$ . *Proc. Amer. Math. Soc.*, 145(6):2407–2409, 2017.
- [Pro10] Yuri G. Prokhorov. Fields of invariants of finite linear groups. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 245–273. Birkhäuser Boston, Inc., Boston, MA, 2010.
- [Sal82] David J. Saltman. Generic Galois extensions and problems in field theory. *Adv. in Math.*, 43(3):250–283, 1982.
- [Sal84] David J. Saltman. Noether's problem over an algebraically closed field. *Invent. Math.*, 77(1):71–84, 1984.
- [Swa69] Richard G. Swan. Invariant rational functions and a problem of Steenrod. *Invent. Math.*, 7:148–158, 1969.
- [Swa83] Richard G. Swan. Noether's problem in Galois theory. In *Emmy Noether in Bryn Mawr (Bryn Mawr, Pa., 1982)*, pages 21–40. Springer, New York-Berlin, 1983.
- [Tre14] Andrey S. Trepalin. Rationality of the quotient of  $\mathbb{P}^2$  by finite group of automorphisms over arbitrary field of characteristic zero. *Cent. Eur. J. Math.*, 12(2):229–239, 2014.
- [Tre16a] Andrey Trepalin. Quotients of conic bundles. *Transform. Groups*, 21(1):275–295, 2016.
- [Tre16b] Andrey Trepalin. Quotients of cubic surfaces. *Eur. J. Math.*, 2(1):333–359, 2016.
- [Yam12] Aiichi Yamasaki. Negative solutions to three-dimensional monomial Noether problem. *J. Algebra*, 370:46–78, 2012.

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