TO TORSION OF ELLIPTIC CURVES AND UNLIKELY INTERSECTIONS

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Abstract. We study effective versions of unlikely intersections of images of torsion points of elliptic curves on the projective line.

To Nigel Hitchin, with admiration.

INTRODUCTION

Let $k$ be a field of characteristic $\neq 2$ and $\bar{k}/k$ an algebraic closure of $k$. Let $E$ be an elliptic curve over $k$, presented as a double cover
$$\pi : E \to \mathbb{P}^1,$$
ramified in 4 points, and $E[\infty] \subset E(\bar{k})$ the set of its torsion points. In [1] we proved:

Theorem 1. If $E_1, E_2$ are nonisomorphic elliptic curves over $\bar{Q}$, then
$$\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$$
is finite.

Here, we explore effective versions of this theorem, specifically, the size and structure of such intersections (see [5] for an extensive study of related problems). We expect the following universal bound:

Conjecture 2 (Effective Finiteness–EFC-I). There exists a constant $c > 0$ such that for every pair of nonisomorphic elliptic curves $E_1, E_2$ over $\mathbb{C}$ we have
$$\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty]) < c.$$

We say that two subsets of the projective line
$$S = \{s_1, \ldots, s_n\}, \quad S' := \{s'_1, \ldots, s'_n\} \subset \mathbb{P}^1(\bar{k})$$
are projectively equivalent, and write $S \sim S'$, if there is a $\gamma \in \text{PGL}_2(\bar{k})$ such that (modulo permutation of the indices) $s_i = \gamma(s'_i)$, for all $i$.

Let $E$ be an elliptic curve over $k$, $e \in E$ the identity, and
$$\begin{align*}
E & \xrightarrow{e} E \\
x & \mapsto -x
\end{align*}$$

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the standard involution. The corresponding quotient map
\[ \pi : E \to E/\iota = \mathbb{P}^1 \]
is ramified in the image of the 2-torsion points of \( E(\bar{k}) \). Conversely, for
\[ r := \{r_1, r_2, r_3, r_4\} \subset \mathbb{P}^1(\bar{k}), \]
the double cover
\[ \pi_r : E_r \to \mathbb{P}^1 \]
with ramification in \( r \) defines an elliptic curve; given another such \( r' \), the curves \( E_r \) and \( E_{r'} \) are isomorphic (over \( k \)) if and only if \( r \sim r' \); in particular, the image of 2-torsion determines the elliptic curve, up to isomorphism.

Let \( E_r[\mathbb{n}] \subset E_r(\bar{k}) \) be the set of elements of order exactly \( \mathbb{n} \), for \( \mathbb{n} \in \mathbb{N} \). The behavior of torsion points of other small orders is also simple:
\[ \pi_r(E_r[3]) \sim \{1, \zeta_3, \zeta_3^2, \infty\}, \]
where \( \zeta_3 \) is a nontrivial third root of 1, and
\[ \pi_r(E_r[4]) \sim \{0, 1, -1, i, -i, \infty\}. \]
In particular, up to projective equivalence, these are independent of \( E_r \).
However, for all \( n \geq 5 \), the sets \( \pi_r(E_r[n]) \), modulo \( \text{PGL}_2(\bar{k}) \), do depend on \( E_r \), and it is tempting to inquire into the nature of this dependence.

In this note, we study \( \pi_r(E_r[n]) \), for varying curves \( E_r \) and varying \( n \). Our goal is to establish effective and uniform finiteness results for intersections
\[ \pi_r(E_r[n]) \cap \pi_{r'}(E_{r'}[n']), \quad n, n' \in \mathbb{N}, \]
for elliptic curves \( E_r, E_{r'} \), defined over \( k \). We formulate several conjectures in this direction and provide evidence for them.

The next step is to ask: given elliptic curves \( E_r, E_{r'} \) over \( \bar{k} \), when is
\[ r \subset \pi_{r'}(E_{r'}[\infty])? \]
We modify this question as follows: Which minimal subsets \( \tilde{L} \subset \mathbb{P}^1(\bar{k}) \) have the property
\[ r \subset \tilde{L} \implies \pi_r(E_r[\infty]) \subseteq \tilde{L}? \]
The sets \( \tilde{L} \) carry involutions, obtained from the translation action of the 2-torsion points of \( E \) on \( E \), which descends, via \( \pi \), to an action on \( \mathbb{P}^1 \) and defines an embedding of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \text{PGL}_2(\bar{k}) \). It is conjugated to the standard embedding of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), generated by involutions
\[ z \mapsto -z \quad \text{and} \quad z \mapsto 1/z, \]
acting on \( \tilde{L} \). This observation is crucial for the discussion in Section 4, where we prove that, modulo projectivities, \( L := \tilde{L} \setminus \{\infty\} \) are fields.
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1. Generalities

Let \( j : E \to \mathbb{P}^1 \) be the standard universal elliptic curve, with \( j \) the \( j \)-invariant morphism. Consider the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\iota} & P \\
\subset & \subset & \\
\mathcal{E} & \xrightarrow{\iota} & \mathcal{P} \\
\downarrow j & & \downarrow j \\
\mathbb{P}^1 & \xrightarrow{\iota} & \mathbb{P}^1
\end{array}
\]

assigning to each fiber \( E_\lambda := j^{-1}(\lambda) \) the quotient \( P_\lambda = \pi(E_\lambda) \simeq \mathbb{P}^1 \), by the involution \( \iota : x \mapsto -x \) on \( E_\lambda \). (This is well-defined even for singular fibers of \( j \).)

Note that \( \mathcal{P} \to \mathbb{P}^1 \) is a \( \text{PGL}_2 \)-torsor. Taking fiberwise \( n \)-symmetric product:

\( P_\lambda \mapsto \text{Sym}^n(P_\lambda) \)

we have associated \( \text{PGL}_2 \)-torsors

\( j_n : \mathcal{P}_n = \text{Sym}^n(\mathcal{P}) \to \mathbb{P}^1. \)

Taking \( \text{PGL}_2 \)-invariants, we have a canonical projection

\( \text{Sym}^n(P_\lambda) \to \mathcal{M}_{0,n}(P_\lambda) \simeq \mathcal{M}_{0,n}, \)

to the moduli space of \( n \)-points on \( \mathbb{P}^1 \). The associated \( \text{PGL}_2 \)-torsor is trivial; fixing a trivialization we obtain a morphism

\( \mu_n : \mathcal{P}_n \to \overline{\mathcal{M}}_{0,n} \)

For every \( N \in \mathbb{N} \), we have the modular curve \( X(N) \to \mathbb{P}^1 \), parametrizing pairs of elliptic curves together with \( N \)-torsion subgroups. The involution \( \iota \) induces an involution on every \( X(N) \), we have the induced quotient

\( X(N) \to Y(N) := X(N)/\iota. \)

Since the family \( j : \mathcal{E} \to \mathbb{P}^1 \) has maximal monodromy \( \text{SL}_2(\mathbb{Z}) \), the curves \( X(N) \) and \( Y(N) \) are irreducible. We have a natural embedding
Y(N) ↣ P. Put

\[ Y := \bigcup_{N \in \mathbb{N}} Y(N) \]

and consider

\[ \text{Sym}^n(Y) \hookrightarrow P_n \rightarrow \mathcal{M}_{0,n}. \]

Note that \( \text{Sym}^n(Y) \) is a union of infinitely many irreducible curves, each corresponding to an orbit of the action of the monodromy group \( \text{PGL}_2(\mathbb{Z}) \) on the generic fiber of the restriction of \( j_n \) to \( \text{Sym}^n(Y) \). Let \( Y_{n,\omega} \subset \text{Sym}^n(Y) \) be an irreducible component corresponding to a \( \text{PGL}_2(\mathbb{Z}) \)-orbit \( \omega \) (for the monodromy action, as above). We now formulate conjectures about \( \mu_n \), for small \( n \), which guide our approach to the study of images of torsion points.

**Conjecture 3.** The map

\[ \mu_4 : Y_{4,\omega} \to \mathcal{M}_{0,4} = \mathbb{P}^1 \]

is finite surjective, for all but finitely many \( \omega \).

**Conjecture 4.** The map

\[ (\mu_4, j) : Y_{4,\omega} \to \mathcal{M}_{0,4} \times \mathbb{P}^1 \]

is a rational embedding, for all but finitely many \( \omega \).

**Conjecture 5.** The map

\[ \mu_5 : Y_{5,\omega} \to \mathcal{M}_{0,5} \]

is a rational embedding, for all but finitely many \( \omega \). Moreover, if for some distinct orbits \( \omega \) and \( \omega' \) the corresponding images \( \mu_5(Y_{5,\omega}) \) and \( \mu_5(Y_{5,\omega'}) \) are curves, then they are different.

**Conjecture 6.** The map

\[ \mu_6 : Y_{6,\omega} \to \mathcal{M}_{0,6} \]

is a rational embedding, for all but finitely many \( \omega \). Moreover, if \( \mu_6(Y_{6,\omega}) \) is a curve then there exist at most finitely many \( \omega' \) such that

- \( \mu_6(Y_{6,\omega'}) \) is a curve and
- \( \mu_6(Y_{6,\omega}) \cap \mu_6(Y_{6,\omega'}) \neq \emptyset. \)

### 2. Examples and evidence

We now discuss examples and evidence for Conjectures in Section 1.

**Example 7.** We have

- \( \mu_4(\text{Sym}^4(Y(2))) \cong \mathcal{M}_{0,4} = \mathbb{P}^1, \)
- \( \mu_4(\text{Sym}^4(Y(3))) \) is a point in \( \mathcal{M}_{0,4} \).
Consider $\text{Sym}^4(Y(4))$. Note that $\pi(E[4]) = \{0, 1, -1, i, -i, \infty\}$ is an orbit of the symmetric group $S_4$, acting on $\mathbb{P}^1$. The pairs $(0, \infty), (1, -1), (i, -i)$ are pairs of stable points for 3 even involutions in $S_4$, and the action of $S_4$ is transitive on pairs and inside each pair. There are two different $S_4$-orbits of 4-tuples: either the orbit contains two pairs of vertices such as $(0, \infty), (1, -1)$, or a pair and two points from different pairs $(0, \infty), (1, i)$. Thus $\text{Sym}^4(Y(4))$ has two components which project to different points modulo $\text{PGL}_2$; therefore, there exist exceptional orbits $\omega$ such that $\mu_4(Y_{4, \omega})$ is a point.

**Lemma 8.** If $\mu_4(Y_{4, \omega})$ is a point then all cross ratios of 4-tuples of points parametrized by $Y_{4, \omega}$ are constant.

**Proof.** The map $\mu_4$ can be viewed as a composition
\[
(P^1)^4 \xrightarrow{\text{cr}} (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \backslash (P^1)^4 / \text{PGL}_2 = P^1 \to S_3 \backslash P^1.
\]
Thus we have a diagram
\[
\begin{CD}
(P^1)^4 @>\text{cr}>> (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \backslash (P^1)^4 / \text{PGL}_2 \\
@VVV @VVV \\
S_4 \backslash (P^1)^4 @>>> S_4 \backslash (P^1)^4 / \text{PGL}_2
\end{CD}
\]
Note that any irreducible $Y_{4, \omega}$ lifts to a union of connected components $Y_{4, \omega, i} \subset (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \backslash Y^4$, where cross-ratio is well defined. Thus if $\mu_4$ is a rational function of cross-ratio on any four-tuple of points and if $\mu_4$ is constant then the cross-ratio is also constant. \hfill \Box

**Proposition 9.** There exist orbits $\omega$ such that $\mu_4 : Y_{4, \omega} \to P^1$ is surjective.

**Proof.** The singular fiber $E_{\infty} := j^{-1}(\infty)$ is an irreducible rational curve with one node $p_\infty$. The group scheme $\bigcup_d \mathcal{E}[d]$, whose generic fiber is isomorphic to $\mathbb{Z}/n \oplus \mathbb{Z}/n$, specializes to $\{\zeta_n\} \subset G_m = E_{\infty} \backslash p_\infty$. Let $E_{\infty}[n]$ be the specialization of $E[n]$; then
- $E_{\infty}[n] \subset \{\zeta_n\}$,
- there exists a subgroup scheme $\mathcal{W}_n \simeq \mathbb{Z}/n \subset \mathbb{Z}/n \oplus \mathbb{Z}/n$ in the group scheme of points killed by $n$, specializing to $E_{\infty}$, while the complementary branches specialize to $p_\infty$.

Taking the quotient by $\iota$, we find that $((\mathbb{Z}/n \oplus \mathbb{Z}/n) \backslash \mathbb{Z}/n) / \iota$ specializes to 0 in the fiber $P^1_\infty$ and all other points specialize to subset in $(\mathbb{Z}/n) / \iota$; the limit depends on the selected direction of specialization.
Assume that we have distinct points \( \{z_1, z_2, z_3, z_4\} \subset \pi(E[n]) \), for a smooth fiber \( E \) of \( \mathcal{E} \), such that \( z_1, z_2 \in W_n/t \) and \( z_3, z_4 \notin W_n/t \).

The \( z_1, z_2 \) can be specialized to different nonzero points in \( \mathcal{E}_\infty/t \), and \( z_3, z_4 \) will specialize to 0.

Assume that \( \mu_4 \) is constant, i.e., the cross-ratio is constant. Since \( z_3, z_4 \) will specialize to 0, the cross-ratio equals 1. Then

\[
(z_1 - z_3)(z_2 - z_4) = (z_2 - z_3)(z_1 - z_4),
\]

and

\[
z_1(z_3 - z_4) = z_2(z_3 - z_4).
\]

Near the special fiber, \( z_3 \neq z_4 \), thus \( z_1 = z_2 \), contradiction. Thus on orbits of this type, \( \mu_4 \) is not constant, hence surjective. \( \square \)

3. Geometric approach to effective finiteness

Let \( E := E_r, E' := E_{r'} \) be elliptic curves. Consider the diagram

\[
\begin{array}{ccc}
C & \to & E \times E' \\
\downarrow & & \downarrow \\
\Delta & \to & \mathbb{P}^1 \times \mathbb{P}^1
\end{array}
\]

where \( C \subset E \times E' \) be the fiberwise product over the diagonal \( \Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \). If \( r \neq r' \) then \( C \) has genus \( \geq 2 \). By Raynaud’s theorem [4],

\[
C(\overline{k}) \cap E[\infty] \times E'[\infty]
\]

is finite, since it is the preimage of \( \pi(E[\infty]) \cap \pi(E'[\infty]) \subset \Delta \), the latter set is also finite. This finiteness argument appeared in [1].

Consider the curves \( C \) occurring in this construction. We have a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\sigma} & E \\
\downarrow & & \downarrow \\
\sigma' & E'
\end{array}
\]

where \( \sigma, \sigma' \) are involutions with fixed points \( c_1, c_2 \) and \( c'_1, c'_2 \), respectively. Assume that

\[
r \cap r' = \{0, 1, \infty\}.
\]

Then the product involution \( \sigma \sigma' \) on \( C \subset E \times E' \) has fixed points in the 6 preimages of the points \( \{0, 1, \infty\} \subset \Delta_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1 \) (diagonally), i.e., is the hyperelliptic involution. Thus there is an action of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) on \( C \), induced by the covering maps \( \pi \) and \( \pi' \). The curve \( C \subset \mathbb{A} = E \times E' \)
has self-intersection $C^2 = 8$ since it is a double cover of both $E$ and $E'$ and its class is equal to $2(E + E')$.

- If the genus $g(C) = 2$ (three such points) then the image of $C$ in its Jacobian $J(C)$ has self-intersection 2. Consider the map
  $$\nu : J(C) \rightarrow A = E \times E'.$$

  and let $n$ be its degree. The preimage $\nu^{-1}(C) \subset J(C)$ has self-intersection $8n$. On the other hand, its homology class is equal to $n$ translations of $C$, hence has self-intersection $2n^2$, thus $n = 4$. Moreover, $\ker(\nu) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by the pairwise differences of preimages of points $\{0, 1, \infty\}$. Thus, $J(C)$ is 4-isogenous to $A := E \times E'$ and $\nu(C)$ is singular, with nodes exactly at the preimages of $\{0, 1, \infty\} \subset \Delta_{\mathbb{P}^1}$. Consider a point $c \in C \subset J(C)$ and assume that $\nu(c)$ has order $m$ with respect to $0 \in A$. Then $c$ has order $m$ or $2m$ in $J(C)$, with respect to $0 \in J(C)$. Hence the corresponding curve $Y(m) \subset \mathbb{P}^1 \times \mathbb{P}^1$ (viewed as a moduli space of pairs $E, E'$) is given as an intersection of genus 2 curves containing a point of order $m$ or $2m$, respectively. This is a locus in the moduli space $\mathcal{M}_2$ of genus 2 curves.

- If $g(C) = 3$ (two such points) then there are three quotients of $C$ which are elliptic curves $E_1, E_2, E_3$, with involutions $\sigma_i \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ fixing 4 points on $E_i$ which are invariant under the hyperelliptic involution given by complement to $\sigma_j$. The kernel of
  $$\nu_i : J(C) \rightarrow E_j \times E_k$$

  contains $E_i$, for $i, j, k \in \{1, 2, 3\}$.

- If $g(C) = 4$ then $C$ is $C/\sigma_i = E_i, i = 1, 2$ and $C/\sigma_1\sigma_2 = C'$ where $g(C') = 2$ and there are exactly two ramification points on $C'$.

- If $g(C) = 5$ then $C/\sigma_1\sigma_2 = C'$ is a hyperelliptic curve of genus 3 and the covering is an unramified double cover.

Remark 10. Assume that there is $b \in \mathbb{P}^1$ and a subset $S \subset C(\bar{k})$ such that $S + b \subset C \subset E \times E'$. Then

$$\#S \leq 8 = C^2 = C \cap (C + b);$$

hence we have at most 8 points $c_i \in \mathbb{P}^1$ such that for $x$-coordinates $c_{i+1}b = c_i + 2b$, where the summation $+1$ corresponds to the summation on the first curve and $+2$ on the second.
Remark 11. The construction can be extended to products of more than two elliptic curves. We may consider

\[ \pi := \prod_{i=1}^{r} \pi_i : A := \prod E_i \to \mathcal{P} := \prod \mathbb{P}_i^1. \]

The ramification divisor of \( \pi : A \to \mathcal{P} \) is a union of products of projective lines. Let \( \Delta = \mathbb{P}_1^1 \subset \mathcal{P} \) be the diagonal, there exists canonical identifications \( \delta_i : \mathbb{P}_i^1 \simeq \Delta \). If \( p \in \Delta \) is contained in \( \delta_i(\pi_i(E_i[\infty])) \), for all \( i \), then the preimage of \( p \) in \( A \) is contained in the preimage of the diagonal. This is a curve of genus at least 2, provided there exist \( E_i, E_j \) with \( r_i \neq r_j \). Then the set of such \( p \) is finite. In particular, if \( E \) is defined over a number field \( k \) and \( p \) is defined over a proper subfield, then \( p \) is also in the image torsion points of \( \gamma(E) \), where \( \gamma \) is a Galois conjugation. Hence, the existence of torsion points with \( x \)-coordinate in a smaller field has a geometric implication.

We expect the following version of Conjecture 2:

**Conjecture 12** (Effective Finiteness–EFC-II). There exists a constant \( c > 0 \) such that for every elliptic curve \( E_r \) over a number field and every \( \gamma \in \text{PGL}_2(\overline{\mathbb{Q}}) \) with \( \gamma(r) \neq r \) we have

\[ \pi_r(E_r[\infty]) \cap \pi_\gamma(E_\gamma[\infty]) < c. \]

4. Fields generated by elliptic division

In this section, we explore properties of subsets of \( \mathbb{P}_1^1(\overline{k}) \) generated by images of torsion points, following closely [1]. For

\[ r := \{r_1, r_2, r_3, r_4\} \subset \mathbb{P}_1^1(\overline{k}), \]

a set of four distinct points, let \( E_r \) be the corresponding elliptic curve defined in the Introduction. Let

\( \tilde{L}_r \subset \mathbb{P}_1^1(\overline{k}) \)

be the smallest subset such that for every \( E_{r'} \) with \( r' \subseteq \tilde{L}_r \) we have \( \pi_{r'}(E_{r'}[\infty]) \subseteq \tilde{L}_r \).

**Theorem 13.** [1] Let \( k \) be a number field. For every \( a \in k \setminus \{0, \pm 1, \pm i\} \), and

\[ r = r_a := \{a, -a, a^{-1}, -a^{-1}\} \subset \mathbb{P}_1^1(k) \]

the set

\[ L_a := \tilde{L}_{r_a} \setminus \{\infty\} \]

is a field.
At first glance, it is rather surprising that such a simple and natural construction, inspired by comparisons of $x$-coordinates of torsion points of elliptic curves, produces a field. The conceptual reason for this is the rather peculiar structure of 4-torsion points of elliptic curves: translations by 2-torsion points yields, upon projection to $\mathbb{P}^1$, two standard commuting involutions on $\mathbb{P}^1(\bar{k})$, which allow to define addition and multiplication on $L_a$.

We may inquire about arithmetic and geometric properties of the fields $L_a$. For $a \in \bar{k}$ we let $k(a) \subseteq \bar{k}$ denote the smallest subfield containing $a$. We have:

- For every $a \in \bar{k}$, the field $L_a$ is a Galois extension of $\mathbb{Q}(a)$.
- For every $k$ of characteristic zero, $L_a$ contains $\mathbb{Q}^{ab}$, the maximal abelian extension of $\mathbb{Q}$.
- The field $L_\zeta$, where $\zeta$ is a primitive root of order 8, is contained in any field $L_a$. Indeed, the corresponding elliptic curve $E$ has ramification subset
  \[ \{\zeta, \zeta^3, \zeta^5, \zeta^7\}, \]
  which is projectively equivalent to $\{1, -1, i, -i\} \subset \pi(E[4])$. Since $\pi(E[4])$ projectively does not depend on the curve $E$, we obtain that $L_\zeta \subset L_a$, for all $a$. The same holds for $L_a$ where $E_a$ is isomorphic to $E_3$ (elliptic curve with an automorphism of order 3).
- The field $L_a$ is contained in a field obtained as an iteration of Galois extensions with Galois groups either abelian or $\text{PGL}_2(\mathbb{F}_q)$, for various prime powers $q$. Is $L_a$ equal to such an extension? As soon as the absolute Galois group is not equal to a group of this type, e.g., for a number field $k$, we have
  \[ L_a \subseteq \bar{k}. \]
- Let $a, a' \in \bar{\mathbb{Q}}$ be algebraic numbers such that $\mathbb{Q}(a) = \mathbb{Q}(a')$. Then $L_a = L_{a'}$. Varying $a \in \mathbb{Q}$, we obtain a supply of interesting infinite extensions $L_a/\mathbb{Q}$.

The rest of this section is devoted to the proof of Theorem 13.

Proof. Let $r_0 := \{0, \infty, 1, -1\}$ and put $L := \tilde{L}_{r_0} \setminus \{\infty\}$. Let $\pi = \pi_{r_a} : E_r \to \mathbb{P}^1$ be the elliptic curve with ramification in $r_a$. Since
  \[ \{0, \infty, \pm 1\} \subseteq \pi(E_r[4]), \]
  we have $L \subseteq L_a$, for all $a$. We first show that $L$ is a field.
Step 1. $L \setminus \{0\}$ is a multiplicative group. Indeed, for any $b \in L \setminus \{0\}$, we have
\[ r_0 := \{0, 1, -1, \infty\} = b^{-1} \cdot \{0, b, -b, \infty\} =: r_b \]
and hence
\[ L_{r_b} = b \cdot L_{r_0} = b \cdot L. \]
Since $b^{-1}, -b^{-1} \in L$ we also have $\{0, 1, -1, \infty\} \subseteq b \cdot L$. Thus $L \subseteq bL$. Similarly, $L \subseteq b^{-1} \cdot L$ or $b \cdot L \subseteq L$, which implies $L = bL$. Thus for any $a, b \in L$ we have $ab \in L$, and since the same holds for $ab^{-1}, b \neq 0$, we obtain $L \setminus \{0\} \subseteq \hat{k}^\times$.

Step 2. Let
\[ \text{Aut}_L := \{ \gamma \in \text{PGL}_2(\hat{k}) \mid \gamma(\tilde{L}) \subseteq \tilde{L} \} \]
be the subgroup preserving $\tilde{L}$. It is nontrivial, since it contains $L \setminus \{0\}$ as a multiplicative subgroup, together with the involution $x \mapsto x^{-1}$.
Consider
\[ \gamma_1 : x \mapsto (x - 1)/(x + 1). \]
It is an involution with $\gamma_1(\infty) = 1, \gamma_1(0) = -1$ and hence $\gamma_1$ is coming from $r := \{0, 1, -1, \infty\}$. Thus it maps $L$ into $L$ and $\gamma_1 \in \text{Aut}_L$.
Consider any pair of distinct elements $\{b, c\} \subseteq L$: it can be transformed into $\{0, 1\}$ by an element from $\text{Aut}_L$. If $b \neq 0, \infty$ then, dividing on $b$, we obtain $\{1, c/b\}$ and $\gamma_1(\{1, c/b\}) = \{0, 1\}$. If $b = 0$ and $c \neq \infty$ then, dividing on $c$, we obtain $\{0, 1\}$. If $b = 0, c = \infty$ then $\gamma_1(\{0, \infty\}) = \{-1, 1\}$ and we reduce to the first case.

Step 3. $L$ is closed under addition. We show that $\gamma : x \mapsto x + 1$ is contained in $\text{Aut}_L$: by Step 2, there exists a $g \in \text{Aut}_L$ which maps $\{-1, \infty\}$ to $\{0, \infty\}$ and hence $\{-1, 0, \infty\}$ to $\{0, b, \infty\}$, for some $b \in L \setminus \{0\}$. Then $b^{-1}g \in \text{Aut}_L$ maps $\{-1, 0, \infty\}$ to $\{0, 1, \infty\}$ and hence $b^{-1}g(x) = \gamma(x) = x + 1$. Thus for any $a \in L$ we have $a + b = b(a/b + 1) \in L$, which shows that $L$ is an abelian group.

Now let us turn to the general $L_a$.

Step 4. Note that $L \subset L_a$ and that $L_a$ is closed under taking square roots. Indeed for any $a \in L$ and $E_r$ with $r := \{0, 1, a, \infty\}$, we have $\sqrt{a} \in \pi_r(E_r[4])$ and hence $\sqrt{a} \in L_a$. Furthermore, for any $a, b \in L_a$ we have $\sqrt{ab} \in L_a$. Indeed, consider the curve $E_r$ with $r = \{0, a, b, \infty\}$. Then $\sqrt{ab} \in \pi(E_r[4])$, since the involution $z \mapsto ab/z$ is contained in the subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ corresponding to the two-torsion on $E_r$, its
invariant points are in $\pi_r(E_r[4])$. Iterating, we obtain that

\[2^{m-1}\sqrt{b_1 \cdots b_m} \in \hat{L}_a \setminus \{\infty\} \text{ for all } b_i \in \hat{L}_a \setminus \{\infty\}\]

**Step 5.** For all $b \in L_a, c \in L$ we have $\sqrt{b+c} \in L_a$. Indeed, for $c \in L$ we know that there is a solution $d \in L$ of the quadratic equation $d^2 + d + c = 0$. Consider the curve $E_r$ for $r := \{\infty, b, d, d + 1\}$. Then

\[d \pm \sqrt{b - d} \in \pi(E_r[4])\]

and hence $d \pm \sqrt{b - d} \in L_a$. Thus

\[\sqrt{(\sqrt{b - d} + d)(\sqrt{b - d} - d)} = \sqrt{b - d^2 - d} = \sqrt{b + c} \in L_a.\]

**Step 6.** Let $P_m \in L[x]$ be a monic polynomial of degree $m$ and let $b \in L_a$. Then there is an $N(m) \in \mathbb{N}$ such that

\[4^{N(m)} \sqrt{P_m(b)} \in L_a.\]

Indeed, we have

\[P_m(b) = c_m + b(c_{m-1} + b(c_{m-2} + \cdots) \cdots).\]

The statement holds for $m = 1$ by Step 4. Assume that it holds for $m - 1$. Then $c_{m-1} + b(c_{m-2} + \cdots) = d 4^{N(m-1)}$ for some $d \in L_a$. We can then write

\[P^m(b) = c_m + bd 4^{N(m-1)},\]

by taking $t = 4^{N(m-1)} \sqrt{b}$ and $u_m = 4^{N(m-1)} \sqrt{c_m}$ we obtain

\[P^m(b) = \prod (t + \zeta^i u_m),\]

where $t \in L_a, u_m \in L$ and $\zeta^i$ runs through the roots of unity of order $4^{N(m-1)}$.

By Steps 4 and 5, we obtain that $4^m 4^{N(m-1)}$-th root of $P_m(b)$ is contained in $L_a$, thus the result holds for $N(m) = 4^{N(m-1)}$.

**Step 7.** Let $b \in L_a$ be any algebraic element over $L$. Then the field $L(b)$ is a finite extension of $L$ and there is an $n \in \mathbb{N}$ such that any $x \in L(b)$ can be represented as a monic polynomial of $b$ with coefficients in $L$ of degree $\leq n$. For such $n$ we define a power $4^N$ such $4^N \sqrt{x} \in L_a$, but then any element in $L(b)$ is in $L_a$. 

\[\square\]
Remark 14. In the proof we have only used points in $\pi(E[4])$. Therefore, for any subset $D \subset \mathbb{N}$ containing 4 we can define $L_{a,D}$, as the smallest subset containing all $\pi(E[n])$ for all $n \in D$ and all elliptic curves obtained as double covers with ramification in $L_{a,D}$. It will also be a field.

For example, if $D = \{3, 4\}$ then $L_{a,D}$ is exactly the closure of $L_a$ under abelian degree 2 and 3 extensions, since $\text{PGL}_2(\mathbb{F}_2) = S_3$ and $\text{PGL}_2(\mathbb{F}_3) = S_4$ and both groups are solvable with abelian quotients of exponent 3, 2.

On $(\text{Sym}^4(\mathbb{P}^1(\overline{\mathbb{Q}})) \setminus \Delta)/\text{PGL}_2(\overline{\mathbb{Q}})$ we can define a directed graph structure $DGS$, postulating that

$$r_z = \{z_1, z_2, z_3, z_4\} \to r_w = \{w_1, w_2, w_3, w_4\}$$

if there is an elliptic curve $E'$ isogeneous to $Er_z$ such that $r_w$ is projectively equivalent to a subset in $\pi(E'[\infty])$. Any path in the graph is equivalent to a path contained in $(\text{Sym}^4(\mathbb{P}^1(L(E))) \setminus \Delta)/\text{PGL}_2(\overline{\mathbb{Q}})$, for some $E$. The graph contains cycles, periodic orbits, and preperiodic orbits, i.e., paths which at some moment end in periodic orbits.

Question 15. Consider the field $L_0 = L_{r_0}$ for $r_0 = \{0, 1, -1, \infty\}$. Does $(\text{Sym}^4(\mathbb{P}^1(L(E))) \setminus \Delta)/\text{PGL}_2(L(E))$ consist of one cycle in $DGS$? Note that any path beginning from $r_0$ extends to a cycle (in many different ways) since $r_0$ is $\text{PGL}_2$-equivalent to a four-tuple of points of order 4 on any elliptic curve.

Remark 16. In Step 7, we have used algebraicity of $L_a/L$, and we do not know how to extend the proof to geometric fields. What are the properties of $L_a$ in geometric situations, when $a$ is transcendental over $k$?

We have seen in the proof that the field $L_a$ is closed under extensions of degree 2. We also have:

Lemma 17. For any $b \in L_a$, we have $\sqrt[3]{b} \in L_a$.

Proof. Consider a curve $E_r$ with $r := \{b, \sqrt{b}, -\sqrt{b}, \infty\}$. Its 3-division polynomial takes the form:

$$f_3(x) = 3x^4 - 4bx^3 - 6bx^2 + 12b^2x - 4b^3 - b^2.$$ 

We can represent it as a product: $3 \prod_{i=1}^{4}(x - x_i)$, where the set $\{x_i\} \subset L_a$ is equal $\pi(E_r[3])$. The corresponding cubic resolvent

$$rc(x) := \prod (x - (x_ix_j + x_kx_l)),$$
where \((i, j), (k, l)\) is any splitting into pairs of indices among 1, 2, 3, 4. In terms of \(b\), we have
\[
rc(x) = x^3 + 2bx^2 + 4b^2x/3 + 8b^3/3 - 128b^4/27 + 64b^5/27.
\]
Since the set \(\{x_i\}\) is projectively equivalent to \(\{0, 1, \zeta_3, \zeta_2^3\}\), we can see that the cubic polynomial above has the form \(C(x^3 + B)\), for some constants \(C, B\). It can be checked that
\[
rc(2b(2x - 1)/3) = (4b/3)^3(x^3 + (b - 1)^2).
\]
After a projective map in \(\text{PGL}_2(L_a)\) we can transform the elements \(x_i x_j + x_k x_l\) into \(-\sqrt[3]{(b - 1)^2}\). Hence \(-\sqrt[3]{(b - 1)^2} \in L_a\), for any \(b \in L_a\); since \(L_a\) is a field closed under 2-extensions we obtain the claim. □

This raises a natural

**Question 18.** Is \(L_a\) is closed under taking roots of arbitrary degree?

If we add \(G_m\) to the set of allowed elliptic curves then the answer is affirmative. However, there may exist a purely elliptic substitute for obtaining roots of prescribed order.

**Corollary 19.** If the \(j(E) \in L_a\) then any set \(\{b, -b, b^{-1}, -b^{-1}\}\) with \(\mu_4((b, -b, b^{-1}, -b^{-1})) = j(E)\) is contained in \(L_a\). Note that such \(b\) are solutions of a cubic equation. Thus \(L_a\) depends only on the curve \(E\) and we will write \(L(E)\).

It is also easy to see that \(L(E) = L(E')\) if \(E\) and \(E'\) are isogenous.

5. INTERSECTIONS

In this section we present further results concerning intersections
\[
\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])
\]
for different elliptic curves \(E_1, E_2\) and provide evidence for the Effective Finiteness Conjecture 2.

**Proposition 20.** Assume that
\[
(2) \quad \pi_1(E_1[4]) = \pi_2(E_2[4]) = \{0, 1, -1, i, -i, \infty\}
\]
and that
\[
\#(\pi_1(E_1[3]) \cap \pi_2(E_2[3])) \geq 2.
\]
Then \(r_1 = r_2\) and \(E_1 = E_2\).
Proof. By our assumption (2), $E_i$ are given by the equation
$$y^2 = x^4 - t_i x^2 + 1.$$ 
With $a_i$ defined by
$$r_i = \{a_i, -a_i, a_i^{-1}, -a_i^{-1}\},$$
we have
$$t_i = a_i^2 + a_i^{-2}.$$ 
We assume that $\pi_i(e_i) = a_i$. In this case, points $\pi_i(E_i[3]) \subset \bar{\mathbb{Q}} \subset \mathbb{P}^1$ are the roots of
$$x^4 + 2a x^3 - (2/a)x - 1 = 0$$
or, equivalently,
$$2x^3 a^2 + (x^4 - 1)a - 2x.$$ 
If $x, y \in \pi_{a_1}(E_{a_1}[3]) \cap \pi_{a_2}(E_{a_2}[3])$, where $x \neq y$ and $a_1 \neq a_2$, then $a_1$ and $a_2$ are the roots of $2x^3 a^2 + (x^4 - 1)a - 2x$ and of $2y^3 a^2 + (y^4 - 1)a - 2y$, that means that their coefficients are proportional
$$\frac{2x^3}{2y^3} = \frac{x^4 - 1}{y^4 - 1} = \frac{-2x}{-2y}.$$ 
Then, on the one hand, $x^3/y^3 = x/y$ implies $x^2 = y^2$, and hence $x = -y$, by our assumption that $x \neq y$. On the other hand,
$$x/y = -1 = (x^4 - 1)/(y^4 - 1) = 1,$$
a contradiction. \qed 

Given any $x \in \bar{\mathbb{Q}}$ we obtain $a_i = a_i(x), i = 1, 2$, which satisfy (3). Then the resulting elliptic curves $E_i$ satisfy (2) and we have
$$\#\{\pi_1(E_1[3]) \cap \pi_2(E_2[3])\} = 1.$$ 
unless
$$(x^4 - 1)^2 + 16x^4 = x^8 + 14x^4 + 1 = 0 \quad \text{or} \quad x^4 = -7 \pm 4\sqrt{3}.$$ 
Moreover,
$$(4) \quad \#\{\pi_{a_1}(E_{a_1}[\infty]) \cap \pi_{a_2}(E_{a_2}[\infty])\} = 6 + 4n \geq 10,$$
where 6 is the number of images of common points of order 4 (from Equation 2) and 4 stands for the size of $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-orbit of a point in $\mathbb{P}^1$. However, it may happen that the inequality in (4) is strict.
Example 21. Consider the polynomial $f_5(x, a)$ defined in [2, Theorem 18]. Its roots are exactly $\pi_a(E_a[5])$. It has degree 12 with respect to $x$ and 6 with respect to $a$. The polynomial $f_3(x, a)$ has degree 2 with respect to $a$ and generically has exactly two solutions $a_1(z), a_2(z)$, for any given $z$. We want also $f_5(v, a_i(z)) = 0$ for some $v$ and $z$. This is equivalent to $f_5(v, a)$ being divisible by $f_3(z, a)$, as polynomials in $a$. Writing division with remainder

$$f_5(v, a) = g(a)f_3(z, a) + C(v, z)a + C'(v, z)$$

for some explicit polynomials $C$ and $C'$, which have to vanish. This condition is gives an explicit polynomial in $u$, which is divisible by a high power of $u$ and $(u - 1)$. Excluding the trivial solutions $u = 0, 1$, and substituting $t = u^4$ we obtain the equation

$$\begin{align*}
32u^{24} + 1369u^{20} + 18812u^{16} + 90646u^{12} + 18812u^8 + 1369u^4 + 32 &= 32t^6 + 1369t^5 + 18812t^4 + 90646t^3 + 18812t^2 + 1369t + 32 \\
t^3\left[32\left(t^3 + \frac{1}{t^3}\right) + 1369\left(t^2 + \frac{1}{t^2}\right) + 18812\left(t + \frac{1}{t}\right) + 90646\right]
\end{align*}$$

Since $t \neq 0$, we have

$$\begin{align*}
&= 32\left(t + \frac{1}{t}\right)^3 + 1369\left(t + \frac{1}{t}\right)^2 + 18716\left(t + \frac{1}{t}\right) + 87908 \\
&= 32r^3 + 1369r^2 + 18716r + 87908 \\
&= f(v)
\end{align*}$$

Computing the discriminant of this cubic polynomial, we find that it has no multiple roots. Its solutions give rise to pairs $u, v$ such that for $a_1 := a_1(u), a_2 := a_2(u)$ we have

$$f_5(v, a_i) = f_3(u, a_i) = 0$$

and hence

$$\#\{\pi_{a_1}(E_{a_1}[\infty]) \cap \pi_{a_2}(E_{a_2}[\infty])\} \geq 14.$$
their common zeroes \((u, v)\) correspond to pairs \((u, v)\) with \(f_3(u, x) = 0\) and \(f_n(v, x) = 0\) simultaneously.

**Example 22.** Applying this scheme to points of order 3 and 7 (or 3 and 11, 3 and 13, 3 and 17) we obtain that the corresponding resultant has roots of multiplicity three which implies the existence of three points \(v\) for a given \(u\) with \(f_3(u, x) = 0\) and \(f_7(v, x) = 0\) simultaneously.

\[
\#\{\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])\} \geq 6 + 16 = 22.
\]

Since we have every reason to expect polynomials \(C_{0,n}(u, v)\) and \(C_{1,n}(u, v)\) to have increasing number of intersection points with the growth of \(n\) we are led to the following conjecture:

**Conjecture 23.** There is an infinite dense subset of points \(a \in \mathbb{P}^1\) such that

\[
\pi_a(E_a[\infty]) \cap \pi_{a_2}(E_{a_2}[\infty]) \geq 14
\]

with

\[
\pi_a(E_a[3]) \cap \pi_{a_2}(E_{a_2}[3]) \neq 0.
\]

Note that in all such cases the fields \(L_a = L_{a_2}\). Numerical evidence suggests that the conjectured inequality may even hold with 22 instead of 14.

6. **General Weierstrass Families**

The family of elliptic curves considered in Section 5 is the most promising for obtaining large intersections of torsion points. In this section, we consider other families where the intersections tend to be smaller, following [2].

We consider elliptic curves \(E_a\) with the same

\[
\pi_a(e_a) = \infty \in \mathbb{P}^1.
\]

These are given by their Weierstrass form

\[
y^2 = x^3 + a_2x^2 + a_4x + a_6.
\]

Using formulas in, e.g., [3, III, Section 2], we write down (modified) division polynomials \(f_{n,a}\), whose zeroes are exactly \(\pi_a(E_a[n])\):

\[
f_{n,a}(x) = \sum_{0 \leq r, s, t, r+2s+3t \leq d(n)} c_{r,s,t}(n) a_2^r a_4^s a_6^t x^{d(n)-(r+2s+3t)},
\]

where \(d(n)\) and the coefficients \(c_{r,s,t}(n)\) can be expressed via totient functions \(J_k(n)\), with \(d(n) = J_2(n)/2\), if \(n > 2\), and \(d(2) = 3\) (see [2]).
Lemma 24. Let $E_1, E_2$ be elliptic curves in generalized Weierstrass form (5) such that, for some $n > 1$ we have

$$\pi_1(E_1[n]) = \pi_2(E_2[n]).$$

Then $E_1 \simeq E_2$.

Proof. The statement is trivial for $n = 2$. For $n > 2$, we have $d(n) \geq 4$, the comparison of division polynomials implies that the terms

$$a^r a^s a^t, \quad r + 2s + 3t \leq 3$$

must be equal. For

$$(r, s, t) = (0, 0, 1), (0, 1, 0), (1, 0, 0)$$

we find equality of coefficients $a_i$ for both curves. \qed

Often, already the existence of nontrivial intersections

$$(6) \quad \pi_1(E_1[n]) \cap \pi_2(E_2[n]) \geq 1$$

leads to the isomorphism of curves $E_1, E_2$. For example, if both curves are defined over a number field $k$ and the action of the absolute Galois group $G_k$ on $\pi_1(E_1[n])$ and $\pi_2(E_2[n])$ is transitive then (6) implies that $E_1 \simeq E_2$. For many, but not all, $n \in \mathbb{N}$, the equality of totient functions $J_2(n) = J_2(m)$, for some $m \in \mathbb{N}$, implies $n = m$.

Example 25. There exist many tuples $(m, n)$ for which

$$J_2(m) = J_2(n) \quad \text{and} \quad J_1(m) \neq J_1(n).$$

For example,

$$J_2(5) = J_2(6) \quad \text{but} \quad J_1(5) = 4, \quad J_1(6) = 2.$$ 

We also have

$$J_2(35) = J_2(40) = J_2(42), \quad \text{while} \quad J_1(35) = 24, J_1(40) = 16, J_1(42) = 12.$$ 

On the other hand, we have

$$J_2(15) = J_2(16) = 192 \quad \text{and} \quad J_1(15) = J_1(16) = 8.$$ 

These results indicate a relation of our question to Serre’s conjecture. He considered the action of the Galois group on torsion points of an elliptic curve $E$ defined over a number field $k$. If $E$ does not have complex multiplication, then the image of the absolute Galois group $G_k$ is an open subgroup of $\text{GL}_2(\hat{\mathbb{Z}})$, i.e., of finite index.

Conjecture 26 (Serre). For any number field $k$ there exists a constant $c = c(k)$ such that for every non-CM elliptic curve $E$ over $k$ the index of the image of the Galois group $G_k$ in $\text{GL}_2(\hat{\mathbb{Z}})$ is smaller than $c$. 
In particular, for $k = \mathbb{Q}$ he conjectured that for primes $\ell \geq 37$ the image of $G_k$ surjects onto $\text{PGL}_2(\mathbb{Z}_\ell)$. Thus, modulo Serre’s conjecture, our conjecture holds for curves defined over $\mathbb{Q}$.

**Proposition 27.** Assume that

$$\pi_1(E_1[n]) = \pi_2(E_2[m]), \quad n \neq m.$$

Then $k(E[n])$ contains $\mathbb{Q}(\zeta_d)$, where $d = \text{lcm}(m, n)$, the least common multiple of $m, n$.

**Proof.** By Serre, we have

$$\mathbb{Q}(\zeta_n) \subset k(E[n]) \quad \text{and} \quad \mathbb{Q}(\zeta_m) \subset k(E[m])$$

as subfields of index at most 2. □

**Corollary 28.** Assume that $k$ does not contain roots of 1 of order divisible by $n, m$. Then $k(E[n]), k(E(m))$ contain a cyclotomic subfield of index at most 2.

This provides a strong restriction on intersections of images of torsion points for elliptic curves over $\mathbb{Q}$, or over more general number fields $k$ with this property. This yields a restriction on fields $k(E[n])$, since $(n, m) > 4$, for all $(n, m)$ with $J_2(n) = J_2(m)$.

**References**


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