CUBIC FOURFOLDS FIBERED IN DEL PEZZO SURFACES OF DEGREE SIX

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The rationality problem for cubic fourfolds has been studied by many authors; see [Has16, §1] for background and references to the extensive literature on this subject. The moduli space $\mathcal{C}$ of smooth cubic fourfolds has dimension twenty. Since the 1990's, the only cubic fourfolds *known* to be rational are:

- the Pfaffian cubic fourfolds and their limits, a divisor $\mathcal{C}_{14} \subset \mathcal{C}$;
- the cubic fourfolds containing a plane $P \subset X$ such that the induced quadric surface fibration $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ has an odd-degree multisection, a countable union of codimension-two loci $\cup \mathcal{C}_K \subset \mathcal{C}$, dense in the divisor $\mathcal{C}_8$ parametrizing cubic fourfolds containing a plane.

Our main result is

Theorem 1. *Let $\mathcal{C}_{18} \subset \mathcal{C}$ denote the cubic fourfolds of discriminant* 18*. There is a countable infinite union of codimension-two loci $\cup \mathcal{C}_K \subset \mathcal{C}$, dense in $\mathcal{C}_{18}$, such that the corresponding cubic fourfolds are rational.*

The main idea is to show that generic cubic fourfolds in $\mathcal{C}_{18}$ admit fibrations in sextic del Pezzo surfaces over $\mathbb{P}^2$, and to characterize which of these are rational over the function field of the base. The corresponding cubic fourfolds are necessarily rational over $\mathbb{C}$.

The first part of the paper introduces the cast of characters: sextic del Pezzo surfaces (Section 1), fourfolds fibered in these surfaces (Section 2), elliptic ruled surfaces of degree six and their connections to cubic fourfolds (Section 3). In Section 4 we construct fibrations in sextic del Pezzo surfaces from elliptic ruled surfaces. The analysis of rationality follows in Section 5. Our approach is grounded in assumptions on the behavior of 'generic' cases; Section 6 validates these in a concrete example. One remaining mystery is the structure of elliptic ruled surfaces of degree six on a cubic fourfold containing such surfaces; Section 7 sheds light on this in a beautiful special case.

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1. Sextic del Pezzo surfaces

Let S be a del Pezzo surface of degree six over a perfect field F. Over the algebraic closure $\bar{F}$, $\bar{S} = S_{\bar{F}}$ is isomorphic to $\mathbb{P}^2$ blown up at three non-collinear points. The lines ((-1)-curves) on $\bar{S}$ consist of the exceptional divisors and the proper transforms of lines joining pairs of the points; these form a hexagon with dihedral symmetry $\mathfrak{D}_{12} \simeq \mathfrak{S}_2 \times \mathfrak{S}_3$. Let $U \subset S$ denote the complement of this hexagon, which is defined over F. We have that $\bar{U} = U_{\bar{F}}$ is an algebraic torus. There exists a torus T over F, classified by the action of $\mathrm{Gal}(\bar{F}/F)$ on the lines, such that U is a principal homogeneous space under T with action defined over F [Man86, Ch. 4,§8].

Proposition 2. [Man66, p. 77] *If $S(F) \neq \emptyset$ then S is rational over F.*

The Galois action on the Picard group gives a representation

$$\rho_S : \mathrm{Gal}(\bar{F}/F) \to \mathfrak{S}_2 \times \mathfrak{S}_3,$$

where the first factor indexes the geometric realizations of S as a blow-up of $\mathbb{P}^2$ and the second factor corresponds to the conic bundle structures. Let K/F denote the quadratic étale algebra associated with first factor and L/F a cubic étale algebra associated with the second factor. Then we have Azumaya algebras B/K and Q/L with the following properties:

- the Brauer-Severi variety $\mathrm{BS}(B)$ has dimension two over K and we have a birational morphism
$$S_L \to \mathrm{BS}(B)$$
realizing S_L as the blow-up over a cycle of three points;
- $\mathrm{BS}(Q)$ has dimension one over L and we have a fibration
$$S_K \to \mathrm{BS}(Q)$$
realizing S_K as a conic fibration with two degenerate fibers;
- the corestrictions $\mathrm{cor}_{K/F}(B)$ and $\mathrm{cor}_{L/F}(Q)$ are split over F;
- B and Q both contain a copy of the compositum KL, and thus are split over this field.

We explain the last condition in geometric terms: $\mathrm{BS}(B)$ admits a distinguished degree-three cycle and the elements of B, interpreted as vector fields over $\mathrm{BS}(B)$ vanishing at this cycle, are isomorphic to KL.

Similarly, $\mathrm{BS}(Q)$ admits a degree-two cycle which yields a copy of KL in Q.

Example 3. It may happen that a del Pezzo surface S has maximal Galois representation ρ_S while the Brauer classes associated with B and Q are trivial. For example, let $F = \mathbb{C}(t)$ and choose a surjective representation

$$\rho : \mathrm{Gal}(\bar{F}/F) \to \mathfrak{S}_2 \times \mathfrak{S}_3$$

corresponding to a branched cover $C \to \mathbb{P}^1$. The geometric automorphism group admits a split exact sequence [Blu10, §2]

$$1 \to T \to \mathrm{Aut}(\bar{S}) \to \mathfrak{S}_2 \times \mathfrak{S}_3 \to 1.$$

Here T is a torus with a natural $\mathfrak{S}_2 \times \mathfrak{S}_3$ action via conjugation. Composing ρ with the splitting, we obtain a del Pezzo surface $S/\mathbb{C}(t)$ of degree six with $\rho_S = \rho$. However, the Brauer group of any complex curve is trivial, so the Brauer classes produced above necessarily vanish.

Proposition 4. *S admits a rational point over F if and only if the Brauer classes B and Q are trivial.* [Cor77]

Corollary 5. *If S admits a zero-cycle of degree prime to six over F then $S(F) \neq \emptyset$.*

Corollary 6. *Let S be as above and assume that Q is trivial as an Azumaya algebra over L. The the following are equivalent:*

- *$S(F) \neq \emptyset$;*
- *S admits a zero cycle of degree prime to three;*
- *B is trivial as an Azumaya algebra over K.*

2. Fibrations over surfaces

A *singular del Pezzo surface* is a surface S with ADE singularities and ample anticanonical class. Those of degree six ($K_S^2 = 6$), are classified as follows [CT88, Prop. 8.3]:

- type I: S also has one A_1 singularity and is obtained by blowing up $\mathbb{P}^2$ in three collinear points, and blowing down the proper transform of the line containing them;
- type II: S has one A_1 singularity and is obtained by blowing up $\mathbb{P}^2$ in two infinitely near points and a third point not on the line associated with the infinitely near points, then blowing down the proper transform of the first exceptional divisor over the infinitely near points;

- type III: S has two A_1 singularities and is obtained by blowing up two infinitely near points and a third point all contained in a line, and blowing down the proper transforms of the first exceptional divisor over the infinitely near point and the line;
- type IV: S has an A_2 singularity and is obtained by blowing up a curvilinear subscheme of length three not contained in a line and blowing down the first two exceptional divisors;
- type V: S has an A_1 and and A_2 singularity and is obtained by blowing up a curvilinear subscheme contained in a line, and blowing down the proper transforms of the first two exceptional divisors and the line.

Types I and II occur in codimension one and correspond to conjugacy classes of involutions associated with the factors of $\mathfrak{S}_2 \times \mathfrak{S}_3$. Type III occurs in codimension two and corresponds to conjugacy classes of Klein four-groups. Type IV also occurs in codimension two and corresponds to three cycles. Type V occurs in codimension three and corresponds to the full group.

Definition 7. Let P be a smooth complex projective surface. A *good del Pezzo fibration* consists of a smooth fourfold $\mathcal{S}$ and a flat projective morphism $\pi : \mathcal{S} \to P$ with following properties:

- fibers of π are singular sextic del Pezzo surfaces of type I, II, III, or IV;
- the curve $B_I \subset P$ parametrizing those of type I is nonsingular;
- the curve $B_{II} \subset P$ parametrizing those of type II is nonsingular away from $B_{IV} \subset P$, the locus of type IV fibers;
- B_{IV} is finite and B_{II} has cusps along B_{IV};
- B_{III} is finite and coincides with the intersection of B_I and B_{II}, which is transverse.

Remark 8. The point is that the classifying map from P is transverse to each singular stratum of the moduli stack of singular del Pezzo surfaces. For example, the discriminant locus is cuspidal at points with A_2 singularity. In particular, good del Pezzo fibrations are Zariski open in the moduli space of all del Pezzo fibrations with fixed invariants.

Proposition 9. *Let $\pi : \mathcal{S} \to P$ be a good del Pezzo fibration. Then Blunk's construction yields:*

- *a non-singular double cover $Y \to P$ branched along B_I;*
- *an element $\eta \in \mathrm{Br}(Y)[3]$;*
- *a non-singular degree-three cover $Z \to P$ branched along B_{II};*
- *an element $\zeta \in \mathrm{Br}(Z)[2]$.*

Proof. As before, let K and L be the quadratic and cubic extensions of $\mathbb{C}(P)$ introduced in Section 1. Let Y and Z denote the normalizations of P in the fields (or étale algebras) K and L respectively.

We first address the double cover. Since B_I is smooth the double cover branched along B_I is also smooth. Consider the base change $\pi_Y : \mathcal{S} \times_P Y$, a sextic del Pezzo fibration with singular fibers of types I, II, III, IV and geometric generic fiber $\overline{S}$. Let G be the Galois group of KL over K. Let $H_1, H_2 \in \mathrm{Pic}(\overline{S})^G$ be the classes corresponding to the birational morphisms

$$\beta_1, \beta_2 : \overline{S} \to \mathbb{P}^2_K.$$

These specialize to Weil divisor classes in each geometric fiber of π_Y. These are Cartier for fibers of types II and IV; H_1 and H_2 are disjoint from the vanishing cycles, reflecting that the resulting divisors are disjoint from the A_1 and A_2 singularities. For fibers of type I, the specializations of H_1 and H_2 coincide and yield smooth Weil divisors containing the A_1 singularity; their order in the local class group is two. For fibers of type III, they contain the A_2 singularity and have order three in the local class group. In each case the resulting curves are parametrized by $\mathbb{P}^2$.

We relativize this as follows: Let $\mathcal{H} \to P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree three. The analysis above shows that its Stein factorization takes the form

$$\mathcal{H} \to Y \to P,$$

where the first morphism is an *étale* $\mathbb{P}^2$-bundle. Thus we obtain the desired class $\eta \in \mathrm{Br}(Y)[3]$.

We turn to the triple cover: Let $\mathcal{H}' \to P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree two. These are fibers of the conic bundle fibrations

$$\gamma_1, \gamma_2, \gamma_3 : \overline{S} \to \mathbb{P}^1_L.$$

A case-by-case analysis shows the conics in type I-IV fibers are still parametrized by $\mathbb{P}^1$'s. Repeating the analysis above, the Stein factorization

$$\mathcal{H}' \to Z \to P$$

consists of an *étale* $\mathbb{P}^1$-bundle followed by a triple cover ramified along B_{II}. The fact that B_{II} has cusps at the points of threefold ramification shows that Z is nonsingular. Indeed, étale locally such covers take the form

$$\mathbb{A}^2 = \{(r_1, r_2, r_3) : r_1 + r_2 + r_3 = 0\} \to \mathbb{A}^2/\{(12)\} \to \mathbb{A}^2/\mathfrak{S}_3$$

where $Z \to P$ corresponds to the second morphism, branched over the discriminant divisor which is cuspidal at the origin. The étale $\mathbb{P}^1$ bundle yields $\zeta \in \mathrm{Br}(Z)[2]$, the desired Brauer class. □

Proposition 10. *Let $\pi : \mathcal{S} \to P$ be a good del Pezzo fibration and fix*

$$\begin{aligned} b_{IV} &= \chi(B_{IV}) = |B_{IV}| \\ b_{III} &= \chi(B_{III}) = |B_{III}| = |B_I \cap B_{II}| \\ b_{II} &= \chi(B_{II} \setminus (B_{III} \cup B_{IV})) \\ b_I &= \chi(B_I \setminus B_{III}), \end{aligned}$$

where χ is the topological Euler characteristic. Then we have

$$\chi(\mathcal{S}) = 6\chi(P) - b_I - b_{II} - 2b_{III} - 2b_{IV}.$$

Proof. This follows from the stratification of the fibration by singularity type. A smooth sextic del Pezzo surface has $\chi = 6$. For types I and II we have $\chi = 5$; for types III and IV we have $\chi = 4$. □

We specialize Proposition 10 to the case where the base is $\mathbb{P}^2$, using Bezout's Theorem and the genus formula:

Corollary 11. *Let $\pi : \mathcal{S} \to \mathbb{P}^2$ be a good del Pezzo fibration; write $d_I = \deg(B_I)$ and $d_{II} = \deg(B_{II})$. Then we have*

$$\chi(\mathcal{S}) = 14 + (d_I - 1)(d_I - 2) + (d_{II} - 1)(d_{II} - 2) - 3b_{IV}.$$

3. Sextic elliptic ruled surfaces and cubic fourfolds

Let E be an elliptic curve and $\mathcal{V} \to E$ be the vector bundle

$$\mathcal{V} = \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are degree-three line bundles. Taking global sections yields

$$\mathcal{O}_E^{\oplus 6} \twoheadrightarrow \mathcal{V}$$

and an embedding

$$T := \mathbb{P}(\mathcal{V}^\vee) \hookrightarrow \mathbb{P}^5$$

as an elliptic ruled surface of degree six. Let $h = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1))$, which corresponds to the restriction of the hyperplane class from $\mathbb{P}^5$. We may identify $\Gamma(\mathcal{O}_{\mathbb{P}^5}(1)) = \Gamma(E, \mathcal{V})$.

Proposition 12. [Hom80, §3] *Let E be a genus one curve and $T \simeq \mathbb{P}(\mathcal{V}^\vee)$ with $\mathcal{V} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles of degree three. Then the resulting $T \hookrightarrow \mathbb{P}^5$ realizes T as an intersection of quadrics and cubics. The defining ideal $\mathcal{I}_T$ has three independent quadratic equations*

$$Q_1 = Q_2 = Q_3 = 0$$

and twenty independent cubic equations.

From now on, assume that $\mathcal{L}_1 \not\simeq \mathcal{L}_2$. We will call $T \subset \mathbb{P}^5$ a *generic decomposable elliptic ruled surface of degree six.*

Remark 13. In the special case where $\mathcal{L}_1 \simeq \mathcal{L}_2$, we may realize

$$T_0 \simeq E \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$$

as a divisor of bidegree $(0,3)$. In this case, the Segre threefold is the locus cut out by the quadratic defining equations of T. We call T_0 a *special decomposable elliptic surface of degree six.*

We return to the generic case. There are two distinguished sections

$$E_1, E_2 \subset T \xrightarrow{p} E$$

corresponding to the rank-one summands. Indeed, write

$$E_1 = h - p^*\mathcal{L}_2, \quad E_2 = h - p^*\mathcal{L}_1$$

which are effective degree-three divisors in T, cut out in $T \subset \mathbb{P}^5$ by three linear equations. We may interpret E_i as the image of E under the line bundle $\mathcal{L}_i$. Let Π_1 and Π_2 denote the planes spanned by E_1 and E_2. The union

$$\Pi_1 \cup_{E_1} T \cup_{E_2} \Pi_2$$

coincides with the locus cut out by the quadratic defining equations by T. Note that this is a complete intersection with trivial dualizing sheaf.

Conversely, a generic decomposable elliptic ruled surface T can be obtained as a residual intersection: Given disjoint planes

$$\Pi_1, \Pi_2 \subset \mathbb{P}^5,$$

consider a generic net of quadrics in their ideal

$$xQ_1 + yQ_2 + zQ_3.$$

Here 'generic' means that the residual subscheme T is smooth and the union

$$\{Q_1 = Q_2 = Q_3 = 0\} = \Pi_1 \cup_{E_1} T \cup_{E_2} \Pi_2$$

is normal crossings. We record some key properties of our construction:

Proposition 14. *The Hilbert scheme of generic decomposable elliptic ruled surfaces* $T \subset \mathbb{P}^5$ *or degree six is isomorphic to a dense open subset of a* $\mathrm{Gr}(3,9)$*-bundle over a dense open subset of* $\mathrm{Gr}(3,6) \times \mathrm{Gr}(3,6)$*. In particular, it is smooth and rational of dimension* 36*.*

Remark 15. The Hilbert scheme of special decomposable elliptic ruled surfaces of degree six is smooth and rational of dimension 33.

Given a generic decomposable sextic elliptic ruled surface $T \subset \mathbb{P}^5$, the ideal $\mathcal{I}_T$ has two cubic generators in addition to Q_1, Q_2, and Q_3. The projective space of cubic fourfolds containing T is of dimension 19.

Proposition 16. *The Hilbert scheme of pairs*

$$\{(T, X) : T \subset X\},$$

where T is a generic decomposable sextic elliptic ruled surface and X is a cubic fourfold, is smooth and rational of dimension 55*.*

Remark 17. The Hilbert scheme of pairs $\{(T_0, X) : T_0 \subset X\}$ with T_0 special is smooth and rational of dimension 52.

The Hilbert scheme of all cubic fourfolds is isomorphic to $\mathbb{P}^{55}$. Forgetting the surface T gives a morphism of 55-dimensional Hilbert schemes; our task it to analyze its fibers.

Proposition 18. *A smooth cubic fourfold containing a sextic elliptic ruled surface contains a family of such surfaces of dimension ≥ 1.*

Proof. The ruled surface T induces an embedding

$$E \hookrightarrow F_1(X)$$

into the Fano variety of lines of X, which is an irreducible holomorphic symplectic manifold. The space of deformations of E as a subscheme of $F_1(X)$ has dimension at least one [Ran95, Cor. 5.1]. On the other hand, any deformations of $F_1(X)$ such that $[E] \in H_2(F_1(X), \mathbb{Z})$ remains algebraic contain deformations of E. □

Remark 19. Proposition 18 may be surprising as $\chi(\mathcal{N}_{T/X}) = 0$ for a sextic elliptic ruled surface T in a cubic fourfold X. But this is more transparent on the holomorphic symplectic variety $F_1(X)$: while $\chi(\mathcal{N}_{E/F_1(X)}) = 0$ the 'expected' dimension of the deformation space of E is one.

There is a direct relation between $\Gamma(\mathcal{N}_{E/F_1(X)})$ and $\Gamma(\mathcal{N}_{T/X})$: Let ℓ be a fiber of $p : T \to E$ and consider the exact sequence

$$0 \to \mathcal{N}_{\ell/T} \to \mathcal{N}_{\ell/X} \to \mathcal{N}_{T/X}|\ell \to 0.$$

Let ℓ vary over E and apply p_*:

$$0 \to \mathcal{T}_E \to \mathcal{T}_{F_1(X)}|E \to p_*\mathcal{N}_{T/X} \to \mathbb{R}^1 p_* p^* \mathcal{T}_E;$$

the last vanishes as p is a $\mathbb{P}^1$-bundle. Thus $p_*\mathcal{N}_{T/X} = \mathcal{N}_{E/F_1(X)}$, inducing the desired isomorphism.

In Section 6, we will show that $h^0(\mathcal{N}_{T/X}) = 1$ generically:

Proposition 20. *Fix an elliptic ruled surface* $T = \mathbb{P}(\mathcal{V}^\vee) \subset \mathbb{P}^5$ *with* $\mathcal{V} = \mathcal{L}_1 \oplus \mathcal{L}_2$, *where* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are generic line bundles of degree three on a generic elliptic curve* E. *Let* X *be a generic cubic fourfold containing* T. *Then the Hilbert scheme parametrizing deformations of* T *in* X *is smooth of dimension one near* T.

The intersection form in the cubic fourfold is [Has00, §4.1]:

	h^2	T
h^2	3	6
T	6	18

Cubic fourfolds admitting algebraic cycles with these intersections form an irreducible divisor $\mathcal{C}_{18}$ in the moduli space $\mathcal{C}$ [Has00, Th. 3.2.3]. Combining with the previous parameter counts (Proposition 16), we obtain

Corollary 21. *Consider the locus of cubic fourfolds containing a decomposable sextic elliptic ruled surface as above. Its closure is an irreducible divisor in the moduli space* $\mathcal{C}$, *the divisor* $\mathcal{C}_{18}$ *parametrizing special cubic fourfolds of discriminant* 18.

4. Constructing the del Pezzo fibration

The fibrations are obtained from sextic elliptic ruled surfaces via residuation:

Proposition 22. *Let* X *be a generic cubic fourfold containing a decomposable sextic elliptic ruled surface* T. *For a generic pencil of quadrics* $L \subset \Gamma(\mathcal{I}_T(2))$, *the intersection*

$$\cap_{\ell \in L} Q_\ell \cap X = T \cup_{D_L} S_L,$$

where S_L *is a projection of a del Pezzo surface of degree six and* $D_L := T \cap S_L$ *is a smooth curve of degree* 12 *and genus* 7. *Varying* L, *we obtain a fibration*

$$\pi : \tilde{X} := \mathrm{Bl}_T(X) \to \mathbb{P}(\Gamma(\mathcal{I}_T(2))^\vee) \simeq \mathbb{P}^2.$$

The preimages of the curves $E_1, E_2 \subset T$ *induce trisections of* π.

Proof. Consider first a generic pencil of quadric hypersurfaces $\{Q_\ell\}_{\ell \in L}$ containing disjoint planes $\Pi_1, \Pi_2 \subset \mathbb{P}^5$, with base locus Y_L. Computing partial derivatives, we see that Y_L has ordinary double points

$$s_{11}, s_{12}, s_{13} \in \Pi_1, \quad s_{21}, s_{22}, s_{23} \in \Pi_2;$$

see also [Kap09, 2.1]. Intersecting with a third generic quadric containing these planes, we obtain

$$\Pi_1 \cup_{E_1} T \cup_{E_2} \Pi_2,$$

where $s_{11}, s_{12}, s_{13} \in E_1$ and $s_{21}, s_{22}, s_{23} \in E_2$. As before, T is a decomposable sextic elliptic ruled surface with distinguished sections

$$E_1, E_2 \subset T \xrightarrow{p} E$$

spanning Π_1 and Π_2 respectively.

Now suppose we intersect Y_L with a generic *cubic* hypersurface X containing T. Let D_L denote the singular locus of $X \cap Y_L$. An application of the Bertini Theorem shows that $D_L \subset T$; similarly, the six singular points $\{s_{ij}\} \in D_L$. Let S_L denote the residual subscheme to T in $X \cap Y_L$. Under our genericity assumptions, S_L and D_L are smooth, and S_L and T intersect in normal crossings along D_L.

The union $W := T \cup_{D_L} S_L$ is a complete intersection of a cubic and two quadrics, thus its dualizing sheaf equals $\mathcal{O}(1)$. The adjunction formula on T allows us to read off the genus and degree of D_L. We have $K_T = -E_1 - E_2$ and $h = K_T + D_L$, thus

$$hD_L = 12, \quad D_L^2 + K_T D_L = 12, \quad D_L E_i = 3;$$

here h is the hyperplane class on T. Moreover, we have

$$\mathcal{O}(1)|S_L = \omega_W|S_L = \omega_{S_L}(D_L).$$

Let H denote the hyperplane class of S_L. We have $HK_{S_L} = -6$ which means hyperplane sections of S_L are genus one curves. It also follows that no multiple $NK_{S_L}, N > 0$, is effective, so S_L is birationally ruled over a curve of genus zero or one. Indeed, since S_L has sectional genus one it is necessarily projectively embedded as an elliptic ruled surface or a sextic del Pezzo [AS90, Th. A].

We use the residuation relations

$$\mathcal{I}_{T \subset W} = \mathcal{H}\mathrm{om}_W(\mathcal{O}_{S_L}, \mathcal{O}_W), \quad \mathcal{I}_{S_L \subset W} = \mathcal{H}\mathrm{om}_W(\mathcal{O}_T, \mathcal{O}_W),$$

and Serre duality to relate invariants of T and S_L:

$$\begin{aligned} H^1(\mathcal{O}_{S_L}) &= \ker(H^2(\mathcal{I}_{S_L}) \to H^2(\mathcal{O}_W)) \\ &= \ker(H^2(\mathcal{H}\mathrm{om}_W(\mathcal{O}_T, \mathcal{O}_W) \to H^2(\mathcal{O}_W)) \\ &= \ker(\mathrm{Ext}^2_W(\mathcal{O}_T, \mathcal{O}_W) \to \mathrm{Ext}^2_W(\mathcal{O}_W, \mathcal{O}_W)) \\ &= \mathrm{coker}(\Gamma(\omega_W) \to \Gamma(\omega_W \otimes \mathcal{O}_T)). \end{aligned}$$

We have already mentioned [Hom80, §3] that T is linearly normal, thus $H^1(\mathcal{O}_{S_L}) = 0$ and S_L is del Pezzo. Symmetrically,

$$H^1(\mathcal{O}_T) = \mathrm{coker}(\Gamma(\omega_W) \to \Gamma(\omega_W \otimes \mathcal{O}_{S_L}))$$

is one-dimensional as T is an elliptic ruled surface. This explains why S_L fails to be linearly normal, obtained via projection from a sextic del Pezzo surface naturally sitting in $\mathbb{P}^6$.

Let $Z_1, Z_2 \subset \tilde{X}$ denote the preimages of E_1 and E_2 respectively; the induced $Z_i \to E_i$ is a $\mathbb{P}^1$-bundle. Let $\tilde{S}_L \subset \tilde{X}$ denote the proper transform of S_L. We have

$$(Z_i \tilde{S}_L)_{\tilde{X}} = (D_L E_i)_{\tilde{T}} = 3$$

hence Z_i is a trisection, as claimed. □

Definition 23. A *fine cubic fourfold of discriminant* 18 is one admitting a fibration satisfying the conclusions of Proposition 22 for some sextic elliptic ruled surface T. A labelling of such a cubic fourfold is a choice of lattice

$$\langle h^2, T\rangle = \langle h^2, S\rangle \subset H^4(X, \mathbb{Z})$$

associated with such a fibration. A cubic fourfold of discriminant 18 is *good* if it is fine and the associated del Pezzo fibration is good.

Since $\tilde{X} = \mathrm{Bl}_T(X)$ and $\chi(T) = 0$, we have

$$\chi(\tilde{X}) = \chi(X) = 1 + 1 + 23 + 1 + 1 = 27.$$

Corollary 11 yields:

Corollary 24. *Let $\tilde{X} \to \mathbb{P}^2$ denote a fibration arising from a good cubic fourfold of discriminant* 18*. Then we have*

$$13 = (d_I - 1)(d_I - 2) + (d_{II} - 1)(d_{II} - 2) - 3b_{III}. \tag{4.1}$$

Proposition 25. *Let $\tilde{X} \to \mathbb{P}^2$ be a del Pezzo fibration, as constructed in Proposition 22. Let $T \to E$ denote the elliptic surface and $\zeta : Z_1 \to \mathbb{P}^2$ the trisection associated with $E_1 \subset T$. Then the morphism ζ is branched over the dual curve to the image of*

$$\phi : E \hookrightarrow \mathbb{P}^2, \quad \phi^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{L}_1^2 \otimes \mathcal{L}_2^{-1},$$

a sextic curve B_{II} with nine cusps.

Proof. As L varies in $\mathbb{P}(\Gamma(\mathcal{I}_T(2))^\vee)$, D_L moves in a linear system on T. As we saw in the proof of Proposition 22, $D_L = h + E_1 + E_2$ meets E_i in degree three, precisely

$$D_L | E_1 = 2[\mathcal{L}_1] - [\mathcal{L}_2].$$

The branch locus of ζ corresponds to elements of this linear series tangent to $\phi(E)$, which is just the dual curve. □

Corollary 26. *In equation (4.1) we have $d_{II} = 6$, $b_{III} = 9$, and thus $d_I = 6$, so B_I is a smooth plane sextic.*

5. Rationality construction

Proposition 27. *Let X be a fine cubic fourfold of discriminant 18 with fibration*

$$\pi : \tilde{X} \to \mathbb{P}^2$$

in sextic del Pezzo surfaces. Let $Y \to \mathbb{P}^2$ denote the associated double cover and $\eta \in \mathrm{Br}(Y)[3]$ the Brauer class. Then the following conditions are equivalent:

- *the generic fiber of π is rational;*
- *the class $\eta = 0$.*

This follows immediately from Corollary 6 and Proposition 2.

Theorem 28. *There exists a dense countable collection of divisors*

$$\mathcal{C}_K \subset \mathcal{C}_{18}$$

parametrizing rational cubic fourfolds.

Proof. We are interested in cubic fourfolds of discriminant 18 with del Pezzo fibration $\tilde{X} \to \mathbb{P}^2$ admitting an algebraic cycle of relative degree coprime to six. Let h and S denote the hyperplane class and the class of the fiber respectively. The description in Proposition 22 means it suffices to find an algebraic cycle Σ with $\Sigma \cdot S$ coprime to 6; and since we have trisections, finding Σ with $Z \cdot S$ coprime to 3 would suffice. Since the integral Hodge conjecture holds for cubic fourfolds [Voi13, Th. 1.4], we need only produce a Hodge class of this type.

For classification purposes, we may assume that the desired Hodge class Σ is the only additional class and meets S in degree 1.

Consider positive definite rank-three overlattices

$$\begin{array}{c|cc} & h^2 & S \\ \hline & 3 & 6 \\ & 6 & 18 \end{array} \subset K_{a,b} := \begin{array}{c|ccc} & h^2 & S & \Sigma \\ \hline h^2 & 3 & 6 & a \\ S & 6 & 18 & 1 \\ \Sigma & a & 1 & b \end{array}$$

with discriminant

$$\Delta = -3 + 12a - 18a^2 + 18b.$$

The lattice $\langle h^2, \Sigma\rangle \cap L^\circ := {h^2}^\perp \subset L$ is even if and only if $a \equiv b \pmod 2$. We assume this parity condition from now on.

Nikulin's results on embeddings of lattices [Nik79, §1.14] imply that the embedding

$$\langle h^2, S\rangle \hookrightarrow L \simeq H^4(X, \mathbb{Z})$$

extends to an embedding of $K_{a,b}$ in L. Replacing Σ with $\Sigma + m(3h^2 - S)$ for a suitable $m \in \mathbb{Z}$, we may assume that $a = -1, 0, 1$. Thus positive integers $\Delta \equiv 9 \pmod{12}$ arise as discriminants, each for precisely one lattice, denoted K_Δ from now on.

Excluding finitely many small Δ, K_Δ defines a divisor

$$\mathcal{C}_{K_\Delta} \subset \mathcal{C}_{18}.$$

See [Has16, §2.3] for details on which Δ must be excluded. These parametrize rational cubic fourfolds.

The density in the Euclidean topology follows from the Torelli Theorem [Voi86] and [Voi07, 5.3.4]. □

Proposition 29 (Hodge-theoretic interpretation)**.** *Let X be a labelled fine cubic fourfold of discriminant* 18*, Λ the Hodge structure on the orthogonal complement of the labelling lattice. Then there exists an embedding of polarized Hodge structures*

$$\Lambda(-1) \hookrightarrow H^2_{prim}(Y, \mathbb{Z}),$$

where (Y', f) is a polarized K3 surface of degree two, and Λ is an index-three sublattice expressible as

$$\Lambda(-1) = {\eta'}^{\perp},$$

where $\eta' \in H^2(Y, \mathbb{Z}/3\mathbb{Z})/\langle f \rangle$ is isotropic under the intersection form modulo 3*.*

This follows from Theorem 9 of [MSTVA14] and the fact that the discriminant group of cubic fourfolds of discriminant 18 is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

Remark 30. We expect that the (Y', η') defined lattice-theoretically coincides up to sign with the (Y, η) arising from the del Pezzo fibration.

6. An explicit example

The computations below were verified symbolically with MAGMA [BCP97] and Macaulay 2 [GS].

Let $\mathbb{P}^5 = \mathrm{Proj}(\mathbb{F}_5[x_0, \dots, x_5])$. Define quadrics

$$Q_1 = 3x_0x_3 + 2x_0x_5 + 4x_1x_3 + 2x_1x_4 + x_2x_3 + x_2x_4 + 2x_2x_5;$$
$$Q_2 = x_0x_3 + 2x_0x_5 + x_1x_3 + 3x_1x_5 + 2x_2x_4 + 3x_2x_5;$$
$$Q_3 = 2x_0x_4 + x_0x_5 + x_1x_3 + 2x_1x_5 + 4x_2x_3 + 3x_2x_5.$$

One can check that the net of quadrics $xQ_1 + yQ_2 + zQ_3$ contains the disjoint planes

$$\{x_0 = x_1 = x_2 = 0\} \qquad \text{and} \qquad \{x_3 = x_4 = x_5 = 0\}.$$

The sextic elliptic ruled surface T, obtained by saturating the ideal of the net of quadrics by the defining ideals of the planes is given by

$$\begin{aligned}
x_0x_4 + 4x_1x_4 + 3x_1x_5 + 4x_2x_3 &= 0,\\
x_0x_4 + x_0x_5 + x_1x_3 + x_1x_4 + 4x_1x_5 + 3x_2x_5 &= 0,\\
x_0x_3 + 4x_0x_4 + x_0x_5 + 4x_1x_4 + 4x_1x_5 + 2x_2x_4 &= 0,\\
x_3^3 + 2x_3x_4^2 + x_3x_4x_5 + 4x_3x_5^2 + 4x_4^3 + 4x_5^3 &= 0\\
x_0^3 + 4x_0^2x_1 + x_0^2x_2 + 2x_0x_1^2 + 2x_0x_1x_2 + 4x_0x_2^2 + x_1^3 + 3x_1^2x_2 + x_2^3 &= 0
\end{aligned}$$

Note that $\mathcal{I}_T(2) = \langle Q_1, Q_2, Q_3\rangle$.

This surface is contained in the cubic fourfold $X = \{f = 0\}$, where

$$\begin{aligned}
f := x_0^3 &+ 4x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + 3x_0^2x_4 + 3x_0^2x_5 + 2x_0x_1^2 + 2x_0x_1x_2\\
&+ 4x_0x_1x_3 + 3x_0x_1x_4 + 4x_0x_1x_5 + 4x_0x_2^2 + x_0x_2x_3 + 3x_0x_2x_4\\
&+ 2x_0x_2x_5 + 3x_0x_3^2 + 4x_0x_3x_5 + 4x_0x_4^2 + 2x_0x_4x_5 + x_0x_5^2 + x_1^3\\
&+ 3x_1^2x_2 + 4x_1^2x_3 + x_1x_2x_3 + 3x_1x_2x_4 + 4x_1x_2x_5 + 3x_1x_3^2 + x_1x_3x_4\\
&+ 2x_1x_4^2 + x_1x_4x_5 + 2x_1x_5^2 + x_2^3 + 4x_2^2x_3 + x_2^2x_4 + 4x_2^2x_5\\
&+ 4x_2x_3^2 + 3x_2x_3x_5 + 3x_2x_4^2 + 2x_2x_4x_5 + 4x_2x_5^2 + 4x_3^3 + 3x_3x_4^2\\
&+ 4x_3x_4x_5 + x_3x_5^2 + x_4^3 + x_5^3
\end{aligned}$$

A direct computation of the partial derivatives of f show that X is smooth. The first order deformations of T as a subscheme of X are given by

$$\Gamma(T, \mathcal{N}_{T/X}) = \mathrm{Hom}(\mathcal{I}_T, \mathcal{O}_T);$$

a direct computation (e.g., in Macaulay 2) gives that this is one-dimensional.

The discriminant locus of the map $\tilde{X} := \mathrm{Bl}_T(X) \to \mathbb{P}^2 = \mathbb{P}(\mathcal{I}_T(2)^\vee)$ is a reducible curve of degree 12, with two irreducible components:

$$\begin{aligned}
B_I : x^6 &+ 2x^4y^2 + x^3y^3 + 4x^3y^2z + 2x^3z^3 + 4x^2y^4 + 4x^2y^2z^2\\
&+ 4x^2yz^3 + 4xy^5 + xy^4z + xy^2z^3 + xyz^4 + 2xz^5 + 4y^6\\
&+ 3y^5z + y^3z^3 + y^2z^4 + 4yz^5 = 0,\\
B_{II} : x^6 &+ 2x^5y + 2x^4y^2 + x^4yz + 4x^3y^3 + 3x^3y^2z + 4x^3yz^2 + x^3z^3\\
&+ 3x^2y^4 + 4x^2y^2z^2 + x^2yz^3 + 3x^2z^4 + 3xy^5 + 2xy^4z\\
&+ 3xy^3z^2 + 3xyz^4 + xz^5 + y^5z + 4y^4z^2 + 3y^3z^3\\
&+ 2y^2z^4 + 4yz^5 = 0
\end{aligned}$$

The curve B_I is smooth, and B_{II} has 9 cusps. Their intersection is a reduced 0-dimensional scheme of degree 36 and is thus transverse.

We found explicit equations over a small field of the following geometric objects:

- a elliptic ruled surface $T \subset \mathbb{P}^5$ residual to a pair of disjoint planes in a net of quadric hypersurface—the corresponding Hilbert scheme is rational of dimension 36 (Prop. 14);
- a cubic fourfold $X \supset T$—this Hilbert scheme is rational of dimension 55 (Prop. 16);
- a fibration in sextic del Pezzo surfaces

$$\pi : \tilde{X} = \mathrm{Bl}_T(X) \to \mathbb{P}^2.$$

We choose them with the following properties:

- T and X are smooth;
- the space of first order deformations of T in X is one-dimensional;
- the fibration π is good.

Since the Hilbert schemes are smooth and rational, the equations we write down readily lift to characteristic zero. The properties we stipulate are open and thus hold for any such lift. Thus our computations yield

Proposition 31. *The cubic fourfolds admitting a generic decomposable sextic elliptic surface are dense in $\mathcal{C}_{18}$. The generic cubic fourfold X admitting one such surface T admits a one-parameter family. For the generic (X, T) the resulting fibration in sextic del Pezzo surfaces is smooth.*

This completes the proof of Proposition 20 and Corollary 21.

7. Visualizing one-parameter families of sextic ruled surfaces

Remark 19 gives some insight into why sextic elliptic scrolls $T \subset X$ deform in one-parameter families on the cubic fourfold X. These can be visualized by specializing in the moduli space. This could be used to obtain a more geometrically motivated argument for Proposition 20.

Let $T_0 \simeq E \times \mathbb{P}^1 \subset \mathbb{P}^5$ be a special decomposable elliptic ruled surface of degree six, introduced in Remark 13. Now let X be a smooth cubic fourfold containing T_0; the space of pairs (T_0, X) is 52-dimensional and the space of all cubic fourfolds is 55-dimensional (see Remark 17). Since T_0 has bidegree $(3, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^1$, X also contains a divisor of bidegree $(0, 3)$. Generically, this consists of three disjoint planes

$$\Pi_1, \Pi_2, \Pi_3 \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

Conversely, three disjoint planes

$$\Pi_1, \Pi_2, \Pi_3 \subset \mathbb{P}^5$$

are automatically pairwise isomorphic: projecting from one identifies the other two. Thus we can realize them as fibers in $\mathbb{P}^2 \times \mathbb{P}^1$. Hence any cubic fourfold containing three disjoint planes also contains a trivial elliptic ruled surface T_0 of degree six.

The intersection form on the algebraic classes of X is

$$K_{54} = \begin{array}{c|cccc} & h^2 & \Pi_1 & \Pi_2 & \Pi_3 \\ \hline h^2 & 3 & 1 & 1 & 1 \\ \Pi_1 & 1 & 3 & 0 & 0 \\ \Pi_2 & 1 & 0 & 3 & 0 \\ \Pi_3 & 1 & 0 & 0 & 3 \end{array}.$$

The associated locus in moduli $\mathcal{C}_{K_{54}} \subset \mathcal{C}$ has codimension three.

Let $F_1(X)$ denote its Fano variety of lines, a holomorphic symplectic fourfold, and $\alpha : H^4(X) \to H^2(F_1(X))$ the Abel-Jacobi map; [Has16, §1] for more details. Write $g = \alpha(h^2)$ and $D_i = \alpha(\Pi_i)$, which may be interpreted as the lines incident to Π_i. The Beauville-Bogomolov form may be written

$$M := \begin{array}{c|cccc} & g & D_1 & D_2 & D_3 \\ \hline g & 6 & 2 & 2 & 2 \\ D_1 & 2 & -2 & 1 & 1 \\ D_2 & 2 & 1 & -2 & 1 \\ D_3 & 2 & 1 & 1 & -2 \end{array}.$$

The divisor g corresponds with the polarization induced from the embedding $F_1(X) \hookrightarrow \mathrm{Gr}(2,6)$.

We elaborate the geometry of these divisors: Projecting from Π_i gives a quadric surface bundle

$$\mathrm{Bl}_{\Pi_i}(X) \to P_i \simeq \mathbb{P}^2,$$

inducing a morphism $D_i \to P_i$. The Stein factorization

$$D_i \to S_i \to \mathbb{P}^2$$

is an étale conic bundle followed by a double cover; S_i is a degree two K3 surface. The intersections $D_{ij} := D_i \cap D_j, i \neq j$, are lines in X incident to both Π_i and Π_j; this is a section of both $D_i \to P_i$ and $D_j \to P_j$, so that $S_i \simeq D_{ij} \simeq S_j$. Computing in the cohomology ring of $F_1(X)$, the intersection $D_{123} = D_1 \cap D_2 \cap D_3$ is a genus one sextic curve $E \subset F_1(X)$; these are lines incident to all three planes and may be interpreted as rulings of T_0 above.

We break symmetry to analyze these ruled surfaces explicitly. It is well known (see, for example, [Has16, §1.2]) that a cubic fourfold containing *two* disjoint planes is rational:

$$X \overset{\sim}{\dashrightarrow} P_1 \times P_2 \simeq \mathbb{P}^2 \times \mathbb{P}^2 \simeq \Pi_2 \times \Pi_1.$$

The forward map blows up Π_1 and Π_2; the inverse blows up D_{12}, regarded as a closed subset of $P_1 \times P_2$. Concretely, D_{12} is a complete intersection of forms of bidegree $(1,2)$ and $(2,1)$, and the inverse map is given by forms of bidegree $(2,2)$ vanishing on D_{12}.

Thus we have

$$\mathrm{Pic}(D_{12}) \supset \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & 5 \\ f_2 & 5 & 2 \end{array},$$

where f_1 and f_2 are induced from the hyperplanes of P_1 and P_2. Now X is in the closure of the Pfaffian locus [Has16, Rem. 7], over which the variety of lines is isomorphic to the length-two Hilbert scheme of the associated K3 surface. This specializes to a birational map

$$D_{12}^{[2]} \overset{\sim}{\dashrightarrow} F_1(X),$$

whence

$$\mathrm{Pic}(F_1(X)) \supset \begin{array}{c|ccc} & f_1 & f_2 & \delta \\ \hline f_1 & 2 & 5 & 0 \\ f_2 & 5 & 2 & 0 \\ \delta & 0 & 0 & -2 \end{array},$$

where 2δ is the class on non-reduced subschemes. Moreover, the divisors $\{D_1, D_2\}$ correspond to the $\{f_1 + 2f_2 - 4\delta, 2f_1 + f_2 - 4\delta\}$ and $g = 2(f_1 + f_2) - 5\delta$.

We return to the case where X contains *three* planes. The locus $D_{123} \subset D_{12}$ is an elliptic curve with degree three under each projection. Thus we have

$$\mathrm{Pic}(D_{12}) \supset \begin{array}{c|ccc} & f_1 & f_2 & E \\ \hline f_1 & 2 & 5 & 3 \\ f_2 & 5 & 2 & 3 \\ E & 3 & 3 & 0 \end{array},$$

which has discriminant 54. Note that $D_3 = E - \delta$ extends the identification of bases from the last paragraph.

Regarding E as an element of

$$\mathrm{Pic}(F_1(X)) \simeq \mathrm{Pic}(D_{12}^{[2]}) \simeq \mathrm{Pic}(D_{12}) \oplus \mathbb{Z}\delta,$$

we see

$$\begin{array}{c|cc} & g & E \\ \hline g & 6 & 6 \\ E & 6 & 0 \end{array} \qquad \begin{array}{c|cc} & h^2 & T_0 \\ \hline h^2 & 3 & 6 \\ T_0 & 6 & 18 \end{array}.$$

We are using the fact that E sweeps out the ruling of T_0.

Finally, consider the linear series $|E|$ on D_{12}, an elliptic fibration. Where do these go under the birational parametrization

$$\mathbb{P}^2 \times \mathbb{P}^2 \overset{\sim}{\dashrightarrow} X?$$

Each member $E_t \in |E|$ gives a decomposable elliptic ruled surface $\tilde{T}_t \subset \mathrm{Bl}_{D_{12}}(\mathbb{P}^2 \times \mathbb{P}^2)$ projecting onto $T_t \subset X$. (They need not be products $E_t \times \mathbb{P}^1$.) The residual planes to T_t are just Π_1 and Π_2; indeed, we can regard

$$\Pi_1 \cup_{E_t} T_t \cup_{E_t} \Pi_2 \subset X$$

as degenerate octic K3 surfaces in X. Deforming X so that it contains T_t, but not Π_1 or Π_2, we generic cubic fourfold fourfold of discriminant 18.

It is natural to expect that the T_t we have constructed sweep out the local deformation space of $T_0 \subset X$, i.e., that $h^0(\mathcal{N}_{T_t/X}) = 1$. This may be checked with a computation similar to that of Section 6.

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